Adiabatic theorems with and without spectral gap condition for non-semisimple spectral values

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We present adiabatic theorems with and without spectral gap condition for operators $A(t): D(A(t)) \subset X \to X$ in a Banach space $X$. In the case with spectral gap, the considered spectral values $\lambda(t) \in \sigma(A(t))$ are not required to be semisimple and, in particular, need not be eigenvalues. In the case without spectral gap, the considered eigenvalues $\lambda(t) \in \partial\sigma(A(t))$ are not required to be weakly semisimple. We also discuss adiabatic theorems for operators $A(t)$ with time-dependent domains and an application to operators defined by symmetric sesquilinear forms.

Keywords: Adiabatic theorems with and without spectral gap condition, not necessarily (weakly) semisimple spectral values, time-dependent domains.

1. Introduction

Adiabatic theory for general – as opposed to skew self-adjoint [1–3] – operators is a quite young area of research which emerged in [4] and was developed further in [5–7]. (In the complex time method [8, 9] in the spirit of [10] non-self-adjoint operators appear as well, yet as auxiliary objects.) In general terms, adiabatic theory is concerned with linear operators $A(t): D(A(t)) \subset X \to X$ in a Banach space $X$ over $\mathbb{C}$, spectral values $\lambda(t)$ of $A(t)$ and projections $P(t)$ in $X$ (for $t \in I := [0, 1]$) such that

- $A(t)$ is densely defined closed for every $t \in I$ and the initial value problems

$$x' = A(\varepsilon s)x \quad (s \in [0, \frac{1}{\varepsilon}]) \quad \text{or} \quad x' = \frac{1}{\varepsilon} A(t)x \quad (t \in [0, 1])$$

with initial conditions $x(s_0) = y$ or $x(t_0) = y$ are well-posed on the spaces $D(A(t))$ for every value of the slowness parameter $\varepsilon \in (0, \infty)$ (Condition 1.1).

- $P(t)$ is spectrally related with $\lambda(t)$ in some natural sense (specified below) for every $t \in I$ except possibly for some few $t$ (Condition 1.2).
What one basically wants to know in adiabatic theory is the following: when – under which additional conditions on $A$, $\lambda$, and $P$ – does the evolution $U_\varepsilon$ generated by the operators $\frac{1}{\varepsilon} A(t)$ (Condition 1.1) approximately follow the spectral subspaces $P(t)X$ related to the spectral values $\lambda(t)$ of $A(t)$ (Condition 1.2) as the slowness parameter $\varepsilon$ tends to 0? Shorter and more precisely: when is it true that

$$(1 - P(t))U_\varepsilon(t)P(0) \longrightarrow 0 \quad (\varepsilon \searrow 0)$$

(with respect to a certain operator topology) for all $t \in I$? Adiabatic theorems are, by definition, theorems that give such conditions. We distinguish adiabatic theorems with spectral gap condition (uniform or non-uniform) and adiabatic theorems without spectral gap condition depending on whether $\lambda(t)$ is isolated in $\sigma(A(t))$ for every $t \in I$ (uniformly or non-uniformly) or not.

We present, in this paper, some new adiabatic theorems for general operators $A(t)$ carrying further the adiabatic theory from [5, 7] mainly in two respects:

- first, by no longer requiring the considered spectral values $\lambda(t)$ of $A(t)$ to be (weakly) semisimple
- second, by no longer requiring the domain of $A(t)$ to be time-independent.

We do not strive for the utmost generality and do not give complete proofs here (which can be found in [11]), but instead try to convey the central ideas.

2. Spectral-theoretic preliminaries

In this preliminary section, we identify a natural notion of spectral relatedness – namely, associatedness in the case with spectral gap and weak associatedness in the case without spectral gap – thereby specifying the vague Condition 1.2 above. Suppose that $A : D(A) \subset X \to X$ is densely defined with $\rho(A) \neq \emptyset$ and $\lambda \in \sigma(A)$. If $\lambda$ is isolated in $\sigma(A)$, then a projection $P$ is called associated with $A$ and $\lambda$ iff $P$ commutes with $A$ and $A_P := A|_{PD(A)}$ is bounded such that

$$\sigma(A_P) = \{\lambda\} \quad \text{whereas} \quad \sigma(A_{1-P}) = \sigma(A) \setminus \{\lambda\}.$$  

Additionally, $\lambda$ is called semisimple iff $\lambda$ is a pole of $(\cdot - A)^{-1}$ of order 1. If $\lambda$ is not necessarily isolated in $\sigma(A)$, then a projection $P$ is called weakly associated (of order $m$) with $A$ and $\lambda$ iff $P$ commutes with $A$ and $A_P$ is bounded such that

$$A_P - \lambda$$  

is nilpotent (of order $m$) whereas $A_{1-P} - \lambda$ is injective with dense range.

Additionally, $\lambda$ is called weakly semisimple iff there is a projection $P$ weakly associated with $A$ and $\lambda$ of order 1, that is, $(A_P - \lambda)^1 = 0$. We have the following central properties of associatedness (which are completely well-known) and of weak associatedness (which seem to be new).
• If \( \lambda \) is isolated in \( \sigma(A) \), then there exists a unique projection \( P \) associated with \( A \) and \( \lambda \), and \( P = (2\pi i)^{-1} \int_{\gamma} (z - A)^{-1} dz \) (Riesz projection). If \( P \) is associated with \( A \) and \( \lambda \), and \( \lambda \) is a pole of \((z - A)^{-1}\) of order \( m \), then

\[
P X = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P) X = \text{rg}(A - \lambda)^k \quad (k \geq m).
\]

• In general, there exists no projection \( P \) weakly associated with \( A \) and \( \lambda \), but if it exists it is unique. If \( P \) is weakly associated with \( A \) and \( \lambda \) of order \( m \), then

\[
P X = \ker(A - \lambda)^k \quad \text{and} \quad (1 - P) X = \text{rg}(A - \lambda)^k \quad (k \geq m).
\]

Clearly, associatedness is a completely natural notion of spectral relatedness in the case with spectral gap. In order to see that weak associatedness is equally natural in the case without spectral gap, notice that it is certainly natural to require that \( A_P \) be bounded with \( \sigma(A_P) = \{\lambda\} \) (in other words, that \( A_P - \lambda \) be quasinilpotent) and that \( A_1 - P - \lambda \) be injective – and from this one naturally arrives at weak associatedness by further requiring that \( A_P - \lambda \) be nilpotent (instead of only quasinilpotent) and that \( \lambda \) belong to the continuous spectrum of \( A_1 - P \) (instead of the residual spectrum). We conclude this section with the following remarks.

• If \( A \) is a spectral operator [12] (with spectral measure \( P^A \)) and \( \lambda \in \sigma(A) \) such that for some bounded neighborhood \( \sigma \) of \( \lambda \) the bounded spectral operator \( A|_{P^A(\sigma)X} \) is of finite type, then the projection weakly associated with \( A \) and \( \lambda \) exists and is given by \( P^A(\{\lambda\}) \). In particular, this is true if \( A \) is spectral of scalar type.

• If \( P \) is associated with \( A \) and \( \lambda \), then the dual operator \( P^* \) is associated with \( A^* \) and \( \lambda \). An analogous implication holds for weak associatedness, provided that \( A \) is a semigroup generator and \( X \) is reflexive.

3. Adiabatic theorems . . .

3.1. . . . with spectral gap condition

We begin with an adiabatic theorem with spectral gap condition, which treats uniform and non-uniform spectral gaps alike and which uses the convenient terminology of \( \lambda(\cdot) \) falling into \( \sigma(A(\cdot)) \setminus \{\lambda(\cdot)\} \) at a point \( t_0 \) (which means there is a sequence \( (t_n) \) converging to \( t_0 \) such that \( \text{dist}(\lambda(t_n), \sigma(A(t_n)) \setminus \{\lambda(t_n)\}) \longrightarrow 0 \) as \( n \to \infty \)). It is a quite simple generalization of the adiabatic theorem of Abou Salem [5] covering the case of semisimple spectral values. Its proof rests upon solving an appropriate commutator equation.
Theorem 3.1. Suppose that

- $A(t) : D \subset X \to X$ for every $t \in I$ generates a contraction semigroup and $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$
- $\lambda(t)$ for every $t \in I$ is a spectral value of $A(t)$, $\lambda(.)$ falls into $\sigma(A(.)) \setminus \{\lambda(.)\}$ at only countably many points accumulating at only finitely many points, and $t \mapsto \lambda(t)$ is continuous
- $P(t)$ for every $t \in I \setminus N$ is associated with $A(t)$ and $\lambda(t)$ and $I \setminus N \ni t \mapsto P(t)$ extends to a twice strongly continuously differentiable map on the whole of $I$, where $N$ denotes the set of those points where $\lambda(.)$ falls into $\sigma(A(.) \setminus \{\lambda(.)\}$.

If $V_\varepsilon$ denotes the (adiabatic) evolution system for $\frac{1}{\varepsilon} A + [P', P]$ on $D$, then

$$\sup_{t \in I} \|U_\varepsilon(t) - V_\varepsilon(t)\| = O(\varepsilon) \quad \text{or} \quad o(1) \quad (\varepsilon \searrow 0),$$

depending on whether $N = \emptyset$ (uniform spectral gap) or $N \neq \emptyset$ (non-uniform gap).

**Sketch of proof.** Since the case with non-uniform gap can be reduced by a standard argument to the case of uniform gap, we have only to consider this latter case. Setting

$$B(t) := \frac{1}{2\pi i} \int_{\gamma_t} (z - A(t))^{-1} P'(t)(z - A(t))^{-1} dz$$

with cycles $\gamma_t$ in $\rho(A(t))$ encircling $\lambda(t)$ but not $\sigma(A(t)) \setminus \{\lambda(t)\}$, we easily obtain (central properties of associatedness!) the commutator equation $BA - AB \subset [P', P]$.

With this commutator equation, in turn, we can rewrite the difference $V_\varepsilon - U_\varepsilon$ as

$$V_\varepsilon(t) - U_\varepsilon(t) = \varepsilon U_\varepsilon(t, \tau)B(\tau)V_\varepsilon(\tau)|_{\tau=0}^{\tau=t} - \varepsilon \int_0^t U_\varepsilon(t, \tau)(B'(\tau) + B(\tau)[P'(\tau), P(\tau)])V_\varepsilon(\tau) d\tau,$$

from which the desired conclusion follows by the boundedness of $U_\varepsilon$ and $V_\varepsilon$ in $\varepsilon$. $\Box$

We refer to [11] for simple examples of the above theorem where the previously known adiabatic theorems [5–7] cannot be applied. In particular, these examples show that the considered spectral values $\lambda(t)$ may well be essential singularities of the resolvent of $A(t)$ and need not even be eigenvalues. A similar – but weaker – theorem is a corollary of an adiabatic theorem of higher order by Joye [6] where the operators $A(t)$ have to be required to analytically depend on $t$ and the considered spectral values are required to be poles of the resolvent.

3.2. without spectral gap condition

We now present the main result of this paper: an adiabatic theorem without spectral gap condition for not necessarily weakly semisimple eigenvalues. It generalizes an
adiabatic theorem independently obtained by Avron, Fraas, Graf, Grech [7] and by Schmid [13], which covers the case of weakly semisimple eigenvalues. Its proof rests upon solving an appropriate approximate commutator equation.

**Theorem 3.2.** Suppose that

- $A(t) : D \subset X \to X$ for every $t \in I$ generates a contraction semigroup and $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$.
- $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that $\lambda(t) + \delta e^{i\theta(t)} \in \rho(A(t))$ for every $\delta \in (0, \delta_0]$ and $t \mapsto \lambda(t), e^{i\theta(t)}$ are continuously differentiable
- $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$,

$$\left\| (\lambda(t) + \delta e^{i\theta(t)} - A(t))^{-1}(1 - P(t)) \right\| \leq \frac{M_0}{\delta} \quad (\delta \in (0, \delta_0]),$$

$rk P(0) < \infty$ and $t \mapsto P(t)$ is continuously differentiable.

If $X$ is reflexive, then

$$\sup_{t \in I} \|U_x(t) - V_x(t)\| \to 0 \quad (\varepsilon \searrow 0),$$

whenever the (adiabatic) evolution system $V_x$ for $\frac{1}{\varepsilon}A + [P', P]$ exists on $D$. If $X$ is arbitrary, then $\sup_{t \in I} \|(1 - P(t))U_x(t)P(0)\| \to 0$ as $\varepsilon \searrow 0$.

**Sketch of proof.** We may assume by a standard approximation argument [3] using mollification that $t \mapsto P(t)$ is twice continuously differentiable (in which case the existence of $V_x$ is guaranteed). Setting $m_0 := rk P(0) = rk P(t)$ (indepent of $t$) and

$$B_\delta := \sum_{k=0}^{m_0-1} \left( \prod_{i=1}^{k+1} \mathcal{F}_{\delta_i} \right) P' P (\lambda - A)^k + \sum_{k=0}^{m_0-1} (\lambda - A)^k P P' \left( \prod_{i=1}^{k+1} \mathcal{F}_{\delta_i} \right)$$

where $\delta := (\delta_1, \ldots, \delta_{m_0})$ and $\mathcal{F}_{\delta} := (\lambda + \delta - A)^{-1}(1 - P)$, we obtain from $PX = ker(A - \lambda)^{m_0}$ (central properties of weak associatedness!) the approximate commutator equation $B_\delta A - AB_\delta \subset [P', P] - C_\delta$, where $C_\delta := C^+_\delta - C^-_\delta$ with

$$C^+_\delta := \sum_{k=0}^{m_0-1} \delta_{k+1} \left( \prod_{i=1}^{k+1} \mathcal{F}_{\delta_i} \right) P' P (\lambda - A)^k, \quad C^-_\delta := \sum_{k=0}^{m_0-1} (\lambda - A)^k P P' (1 - \delta_{k+1}) \left( \prod_{i=1}^{k+1} \mathcal{F}_{\delta_i} \right).$$

At first glance, this solution $B_\delta$ of the approximate commutator equation might seem a bit mysterious, but actually, it can be guessed from the solution $B$ of the commutator equation in the case with spectral gap: indeed, if $\lambda$ is a pole of $(\cdot - A)^{-1}$ of order at most $m_0$, then $B$ can be seen to be equal to $B_{\delta=0}$ with the help of Cauchy’s theorem. With the above approximate commutator equation, we can now
rewrite the difference $V_\varepsilon - U_\varepsilon$ as

$$
V_\varepsilon(t) - U_\varepsilon(t) = \varepsilon U_\varepsilon(t, \tau) B_\delta(\tau) V_\varepsilon(\tau) \bigg|_{\tau=0}^{\tau=t} - \varepsilon \int_0^t U_\varepsilon(t, \tau) (B_\delta'(\tau) + B_\delta(\tau)) \cdot [P'(\tau), P(\tau)] V_\varepsilon(\tau) \, d\tau + \int_0^t U_\varepsilon(t, \tau) C_\delta(\tau) V_\varepsilon(\tau) \, d\tau
$$

and with the assumed resolvent estimate we obtain the estimates

$$
\|B_\delta(t)\| \leq c (\delta_1 \cdots \delta_{m_0})^{-1}, \quad \|B_\delta'(t)\| \leq c (\delta_1 \cdots \delta_{m_0})^{-(m_0+1)},
$$

$$
\int_0^1 \|C_\delta^+(\tau)\| \, d\tau \leq c \sum_{k=0}^{m_0-1} (\delta_1 \cdots \delta_k)^{-1} \eta^+(\delta_{k+1})
$$

where $\eta^+(\delta) := \int_0^1 \delta \|R_\delta(\tau) P'(\tau) P(\tau)\| \, d\tau$ and $\eta^-(\delta) := \int_0^1 \delta \|P(\tau) P'(\tau) \overline{R}_\delta(\tau)\| \, d\tau$ for $\delta \in (0, \delta_0]$. With $(1 - P) X = \text{rg}(A - \lambda)^{m_0}$ and $(1 - P^*) X^* = \text{rg}(A^* - \lambda)^{m_0}$ (weak associatedness for semigroup generators in reflexive spaces carries over to the dual operators!) and with the assumed resolvent estimate one can show that

$$
\delta \overline{R}_\delta(t), \quad \delta \overline{R}_\delta(t)^* \longrightarrow 0 \quad (\delta \searrow 0)
$$

w.r.t. the strong operator topology for almost every $t \in I$, and since $\text{rk} P(t)^* = \text{rk} P(t) < \infty$, one even gets $\eta^+(\delta) \longrightarrow 0$. Choosing now $\delta = \delta_\varepsilon = (\delta_1, \ldots, \delta_{m_0})$ in a suitable way, we arrive at the desired conclusion.

We refer to [11] for a wealth of simple examples of the above theorem where the previously known adiabatic theorems [7, 13] cannot be applied (finding interesting applied examples will be the subject of future research). In these examples the operators $A(t)$ need not be spectral, but for spectral operators the construction of examples is particularly simple: one can ensure the assumptions – in particular, the reduced resolvent estimate – of the above theorem by choosing $A(t)$ for every $t$ as a spectral operator (with spectral measure $P^{A(t)}$) generating a contraction semigroup and $\lambda(t)$ as an eigenvalue of $A(t)$ such that, apart from regularity conditions,

- $A(t)|_{P^{A(t)}(\sigma(t))}$ is spectral of scalar type for some punctured neighborhood

$$
\sigma(t) := \sigma(A(t)) \cap \overline{U_{r_0}(\lambda(t))} \setminus \{\lambda(t)\}
$$

of $\lambda(t)$ in $\sigma(A(t))$ ($r_0 \in (0, \infty) \cup \{\infty\}$) and $\text{rk} P^{A(t)}(\{\lambda(t)\}) < \infty$ for a.e. $t$,

- the open sector $\{\lambda(t) + \delta e^{i\theta} : \delta \in (0, \delta_0), \theta \in (\vartheta(t) - \vartheta_0, \vartheta(t) + \vartheta_0)\}$ of radius $\delta_0 \in (0, \infty)$ and angle $2\vartheta_0 \in (0, \pi)$ is contained in $\rho(A(t))$.

In the special case where the eigenvalues $\lambda(t)$ from the above theorem are purely imaginary for every $t \in I$, these eigenvalues are automatically weakly semisimple by the contraction semigroup assumption, and so, in this special case, the adiabatic theorem above reduces to the theorem from [7, 13]. Simple examples [6, 11] show that the contraction semigroup assumption cannot be essentially weakened.
3.3. . . for time-dependent domains

So far, we have always considered operators $A(t)$ with time-independent domains. It is possible to quite easily extend all the presented adiabatic theorems to operators with time-dependent domains, which in essence is due to the following facts:

- An evolution system $W = (W(t, s))$ for operators with time-dependent domains is right-differentiable w.r.t. the second variable $s$ in some appropriate sense.
- A continuous, right-differentiable map $f : [a, b] \to X$ with bounded right derivative $\partial_+ f$ belongs to the Sobolev class $W^{1,\infty}$ and therefore the fundamental theorem of calculus is available.

We do not state these extended adiabatic theorems with their precise assumptions (which can be found in [11]), but instead give an application to the special case of skew self-adjoint operators $A(t) = iA(t)$ defined by symmetric sesquilinear forms $a(t)$ (Schrödinger operators with Rollnik potentials $V(t)$, for instance). Consider the situation, where $H^+$ is continuously and densely embedded in $H$ (both Hilbert spaces), where $a(t)$ is a symmetric sesquilinear form on $H^+$ such that, for some $m \in (0, \infty)$,

$$\langle \cdot, \cdot \rangle_t^+ := a(t)\langle \cdot, \cdot \rangle + m\langle \cdot, \cdot \rangle$$

is a scalar product on $H^+$ and $\| \cdot \|^+_t$ is equivalent to $\| \cdot \|^+$ for every $t \in I$, and where $t \mapsto a(t)(x, y)$ is twice continuously differentiable for all $x, y \in H^+$. We then have the following adiabatic theorem without spectral gap condition, which is a generalization of a theorem of Bornemann [2] where the considered eigenvalues are required to belong to the discrete spectrum. It can be proved in a considerably simpler way than in [2], namely by applying the general adiabatic theorems for time-dependent domains mentioned above.

Theorem 3.3. Suppose that

- $A(t) = iA(t) : D(A(t)) \subset H \to H$ with $a(t)$ as above.
- $\lambda(t)$ for every $t \in I$ is an eigenvalue of $A(t)$ such that $t \mapsto \lambda(t)$ continuous.
- $P(t)$ is weakly associated with $A(t)$ and $\lambda(t)$ for almost every $t \in I$, $\text{rk} P(0) < \infty$ and $t \mapsto P(t)$ is continuously differentiable.

Then

$$\sup_{t \in I} \| (1 - P(t))U_z(t)P(0) \|, \quad \sup_{t \in I} \| P(t)U_z(t)(1 - P(0)) \| \to 0 \quad (\varepsilon \searrow 0).$$
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