Long-Time Asymptotics for the Wave Equation of Linear Elasticity in Cylindrical Waveguides

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Abstract

Let $\Omega = \mathbb{R} \times D$, $D \subset \mathbb{R}^2$ bounded. Suppose that $\Omega$ is filled with a homogeneous and isotropic elastic medium, which is fixed at the boundary, and that a time-harmonic force $f(x)e^{-ikt}$ acts in $\Omega$, where $f$ has bounded support. The resulting elastic motion and its behaviour as $t \to \infty$ is studied. Depending on the choice of $\omega$ and $f$, two different types of time-asymptotic occur: Either the motion is unbounded as $t \to \infty$ at almost every $x \in \Omega$ (resonance case), or the principle of limiting amplitude holds. The resonance is shown to occur for countable many frequencies of incitation. Even two-dimensional elastic motion in a domain $\Omega = \mathbb{R} \times (0, 1)$ is considered; here the same phenomena happen. These results are proved using an explicit representation of the motion, which is obtained combining spectral- and Fourier-transform. The method is presented in a general setting, so that it is applicable also to other translation-invariant wave equations.
1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be filled with a homogeneous and isotropic elastic medium, which is fixed at the boundary $\partial \Omega$. Suppose that a time-harmonic force $F(x) e^{-i\omega t}$ (with $F : \Omega \to \mathbb{R}^3$) acts in $\Omega$. Denote by $s(t, x) \in \mathbb{R}^3$ the resulting motion of a point $x \in \Omega$ at time $t \geq 0$. Then $s$ is solution of (see e.g. [1] or [13])

$$\begin{align*}
\mu \Delta s(t, x) + (\lambda + \mu) \text{grad div } s(t, x) \\
+ \sigma F(t, x) e^{-i\omega t} = \sigma \partial_t^2 s(t, x) & \quad \text{for } t \geq 0, \ x \in \Omega, \\
\ \\
\ s(t, x) = 0 & \quad \text{for } t \geq 0, \ x \in \partial \Omega, \\
\ s(0, x) = s_0(x), \ \partial_t s(0, x) = s_1(x) & \quad \text{for } x \in \Omega,
\end{align*}$$

\[ (1.1) \]

where $\sigma > 0$ (density of mass) and $\mu > 0$, $3\lambda + 2\mu > 0$ (Lamé-constants). More general, we study in a domain $\Omega \subset \mathbb{R}^n$ the solution $u : [0, \infty) \times \Omega \to \mathbb{R}^n$ of the initial-boundary-value problem

$$\begin{align*}
\partial_t^2 u(t, x) - (\Delta + c_0 \text{grad div }) u(t, x) & = f(x) e^{-i\omega t} & \quad \text{for } t \geq 0, \ x \in \Omega, \\
\ u(t, x) = 0 & \quad \text{for } t \geq 0, \ x \in \partial \Omega, \\
\ u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x) & \quad \text{for } x \in \Omega,
\end{align*}$$

\[ (1.2) \]

where $c_0 > -1$ is supposed. The last condition $c_0 > -1$ is needed to guarantee the coerciveness of the spatial operator. It is satisfied in all physically relevant situations, since $c_0 = 1 + \frac{\lambda}{\mu}$. Let $\Omega$ be given by

$$\Omega = \mathbb{R} \times D, \quad D \subset \mathbb{R}^{n-1} \text{ bounded.} \quad (1.3)$$

In the case $n = 3$, if $D$ is connected, then $\Omega$ is an infinite tube with cross-section $D$. If $n = 2$ and $D = (a, b)$ with $-\infty < a < b < \infty$, then $\Omega$ is a layer.

We are interested in the asymptotic behaviour as $t \to \infty$ of $u$ solving (1.2). In order to give a brief description of the results, we suppose that $\partial D \in C^\infty$. It will be proved that there exists a set $\omega_{\text{res}} \subset \mathbb{R}$ of resonance frequencies, which is countable and has no finite accumulation point. If the frequency $\omega$ of incitation coincides with a resonance frequency, then resonance occurs: There exist $N \in \mathbb{N}$,
\( \alpha_1, \ldots, \alpha_{N-1} \in (0, 1) \) and \( f_1, \ldots, f_N, u_\omega \in C(\overline{\Omega}) \), such that
\[
    u(t, x) = \sum_{j=1}^{N-1} e^{-i\omega t} \alpha_j f_j(x) + e^{-i\omega t} \ln t \cdot f_N(x) + e^{-i\omega t} u_\omega(x) + o(1) \quad (1.4)
\]
as \( t \to \infty \), where the precise definition of the meaning of \( o(1) \) is given in Theorem 5.2. The functions \( f_1, \ldots, f_N \) depend on \( f \) and can be computed together with the values of \( \alpha_1, \ldots, \alpha_{N-1} \in (0, 1) \) solving a parameter-dependent eigenvalue problem in \( D \). This will be done for a special example of \( D \) in a subsequent paper. The present article shows, that at least one of \( f_1, \ldots, f_N \) does not vanish, if \( f \) is chosen suitable, and that the corresponding exponent \( \alpha_j \) is element of \( \left[ \frac{1}{2}, 1 \right) \).

If \( \omega \in [0, \infty) \setminus \omega_{\text{res}} \), then \( u \) satisfies the principle of limiting amplitude
\[
    u(t, x) = e^{-i\omega t} u_\omega(x) + o(1) \quad \text{as } t \to \infty, \quad (1.5)
\]
where \( u_\omega \in C^2(\overline{\Omega}) \) solves
\[
\left\{
\begin{array}{l}
    (-\Delta - c_0 \text{grad div} - \omega^2) u_\omega(x) = f(x) \quad \text{in } \Omega, \\
    u(x) = 0 \quad \text{on } \partial \Omega.
\end{array}
\right. \quad (1.6)
\]

These results are obtained using a method, which was developed in [12] and used in [8]. It is based on an explicit representation of the spectral family of the spatial operator in (1.2), which is computed using Fourier-transform with respect to the unbounded variable. This article presents the method in a more general setting, so that it can be applied to other wave equations being translation invariant.

In order to be more precise, let \( \mathcal{H} \) be a Hilbert-space given by
\[
\mathcal{H} := L_2(\mathbb{R}, H), \quad H = \text{separable Hilbert-space} \quad (1.7)
\]
(e.g. \( H = L_2(D)^n \), \( \mathcal{H} = L_2(\Omega)^n \)). The Fourier-transform with respect to the first variable is defined by
\[
\mathcal{F} := F \otimes \text{Id} : \mathcal{H} = L_2(\mathbb{R}) \otimes H \to \mathcal{H} \quad (\otimes = \text{tensor product}), \quad (1.8)
\]
with $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ and $\text{Id} : H \to H$ denoting respectively usual Fourier-transform and identity. Let $\mathcal{A}$ be an operator in $\mathcal{H}$, given by

$$\mathcal{A} = \mathcal{F}^{-1} \circ \int_{\mathbb{R}}^{\oplus} \lambda(\xi) d\xi \circ \mathcal{F},$$

(1.9)

where $\{\lambda(\xi)\}_{\xi \in \mathbb{R}}$ denotes a family of self-adjoint operators in $H$ (for the notation see section XIII.16 of [14]). Roughly spoken, (1.9) means that $\lambda(\xi)$ defines the Fourier-transform of $\mathcal{A}$ by

$$(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f)(\xi) = \lambda(\xi) f(\xi) \quad \text{for almost every } \xi \in \mathbb{R},$$

if $f \in D(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}) \subset L_2(\mathbb{R}, H)$. The operator family is supposed to satisfy the following assumptions:

(A1) For every fixed $\xi \in \mathbb{R}$, $\lambda(\xi)$ is self-adjoint and positive in $H$.

(A2) For every fixed $\xi \in \mathbb{R}$, $\lambda(\xi)$ has an orthonormal system of eigenfunctions $\{\psi_j(\xi)\}_{j \in \mathbb{N}}$ being complete in $H$. The corresponding eigenvalues are denoted by $\lambda_j(\xi)$, where every eigenvalue eventually has to be counted multiple times according to his multiplicity, which is supposed to be finite.

(A3) For every fixed $j \in \mathbb{N}$, $\lambda_j(\xi)$ and $\psi_j(\xi)$ depend analytically on $\xi$.

(A4) For every $\lambda \in \mathbb{R}$, the set $\{(j, \xi) \in \mathbb{N} \times \mathbb{R} : \lambda_j(\xi) = \lambda\}$ is empty or finite, and for every fixed $j \in \mathbb{N}$, $\lambda_j(\xi) \to +\infty$ as $\xi \to \pm\infty$.

Then $\mathcal{A}$ is self-adjoint in $\mathcal{H}$ (see [14], sec. XIII.16).

Section 2 studies the spectral family of $\mathcal{A}$. The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of one infinite interval and is purely absolutely continuous. We say that $\lambda \in \sigma(\mathcal{A})$ is a resonance point, if and only if

$$\exists (j, \xi) \in \mathbb{N} \times \mathbb{R} : \lambda = \lambda_j(\xi) \wedge \frac{d\lambda_j}{d\xi}(\xi) = 0.$$ 

(1.10)

The set of all resonance points is countable and has no finite accumulation point. The spectral family $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ of $\mathcal{A}$ will be shown to have a Hölder-continuous derivative
with respect to $\lambda$ (in a certain sense) at every $\lambda \in \mathbb{R}$ being no resonance point. If $\lambda_0 \in \mathbb{R}$ is a resonance point, then the derivative with respect to $\lambda$ of the spectral family has a singularity of type $|\lambda - \lambda_0|^{-\alpha}$ with $\alpha \in (0, 1)$. For a more precise statement see (2.15).

Section 3 studies the solution of the initial value problem

\[
\begin{align*}
  u & \in C^2([0, \infty), \mathcal{H}), \quad u(t) \in D(A) \quad \text{for } t \geq 0, \\
  u''(t) + Au(t) & = f e^{-i\omega t} \quad \text{for } t \geq 0, \\
  u(0) & = u_0, \quad u'(0) = u_1,
\end{align*}
\]  

(1.11)

where $u'$ denotes the derivative of $u$. The solution $u$ is given by spectral integrals.

Theorem 3.2 shows the connection between the long-time behaviour of $u$ and the properties of the spectral family $\{P_\lambda f\}$ at $\lambda = \omega^2$ and at $\lambda = 0$. If $\omega^2$ is a resonance point and if $f$ is chosen suitable, then $u$ shows resonance. If $\omega^2$ is no resonance point and if the behaviour of the spectral family near $\lambda = 0$ is not too bad, then the principle of limiting amplitude holds.

Section 4 studies spectral properties of the elastic spatial operator. It is shown that the theory developed in Sections 2 and 3 can be applied to problem (1.2). In section 5, the long-time behaviour of the solution $u$ of (1.2) is computed. Application of the general theory yields estimates with respect to some weighted norms. Using elliptic regularity theory, pointwise estimates of $u(t, x)$ as $t \to \infty$ are proved.

Equation (1.1) with other (non homogeneous) boundary conditions in a domain $\Omega = \mathbb{R}^d \times (0, h)$ is considered in the modelling of seismic waves. In this context, L. Brevdo obtained similar resonance effects in [3], [4] and [5]. He studies the solution using spatial Fourier-transform and Laplace-transform with respect to time. This method was developed by R. Briggs in [6]. The theory in this book leads to the same definition of resonance points as (1.10). The considerations there are not restricted to self-adjoint spatial operators. But one has to pay for this generality: It has to be distinguished between resonance points leading to convective instabilities and those leading to absolute instabilities. The result of the present paper shows in the case of a self-adjoint spatial operator, that at every resonance point an absolute instability
I want to thank P. Werner and J. Giannoulis for various discussions. Thanks to S. Müller, who gave me a hint how to optimize the underlying Sobolev-spaces.

2 Spectral properties of \( A \)

We study the spectral family \( \{P_\lambda\} \) of the self-adjoint operator \( A \) given by (1.9). The main result in this section consists of an explicit representation of the spectral family (see Theorem 2.3).

Suppose that the operator family \( \{A(\xi)\}_{\xi \in \mathbb{R}} \) satisfies Assumptions (A1) – (A4) given on page 4. Set

\[
\lambda_{\min} := \min \{\lambda_j(\xi) : j \in \mathbb{N}, \ \xi \in \mathbb{R}\}. 
\] (2.1)

From (A3) and (A4), one obtains that the given set has a minimum. Furthermore, (A1) implies that \( \lambda_{\min} \geq 0 \). Define the set of resonance points by

\[
\sigma_{\text{res}}(A) := \left\{ \lambda_j(\xi) : j \in \mathbb{N} \land \xi \in \mathbb{R} \land \frac{d\lambda_j}{d\xi}(\xi) = 0 \right\}. 
\] (2.2)

By (A3) and (A4), \( \sigma_{\text{res}}(A) \) is countable infinite and has no finite accumulation point. Obviously \( \lambda_{\min} \in \sigma_{\text{res}}(A) \). Write \( \sigma_{\text{res}}(A) \) as

\[
\sigma_{\text{res}}(A) = \{\sigma_1, \sigma_2, \ldots\} \quad \text{with} \quad \lambda_{\min} = \sigma_1 < \sigma_2 < \ldots, \ \sigma_j \to \infty \text{ as } j \to \infty. \] (2.3)

In order to define local inverses of the mapping \( \xi \mapsto \lambda_j(\xi) \), let \( j \in \mathbb{N} \) be fixed. By the analycity of \( \lambda_j \), the set

\[
S_j := \left\{ \xi \in \mathbb{R} : \frac{d\lambda_j}{d\xi}(\xi) = 0 \right\}
\]

is finite or countable with no finite accumulation point. Hence one can write

\[
S_j = \{\rho_{jk} : k \in \mathbb{I}_j\} \quad \text{with} \quad \rho_{jk} < \rho_{j(k+1)} \text{ if } k, k + 1 \in \mathbb{I}_j,
\]
where $\mathbb{I}_j = \{1, \ldots, N\}$ or $\mathbb{I}_j = \mathbb{N}$ or $\mathbb{I}_j = \mathbb{Z}$. Define intervals $I_{jk}$ by

$$I_{jk} := \begin{cases} (-\infty, \rho_j(k+1)) & \text{if } k \not\in \mathbb{I}_j, \ k+1 \in \mathbb{I}_j, \\ [\rho_{jk}, \rho_j(k+1)] & \text{if } k, \ k+1 \in \mathbb{I}_j, \\ [\rho_{jk}, \infty) & \text{if } k \in \mathbb{I}_j, \ k+1 \not\in \mathbb{I}_j \end{cases}$$

(2.4)

for $k \in \bar{\mathbb{I}}_j$, where $\bar{\mathbb{I}}_j := \mathbb{I}_j \cup \{\min\{k \in \mathbb{I}_j\} - 1\}$ if $\mathbb{I}_j$ is bounded from below, and $\bar{\mathbb{I}}_j := \mathbb{I}_j$ otherwise. Then $\{I_{jk}\}_{k \in \bar{\mathbb{I}}_j}$ defines a decomposition of $\mathbb{R}$ having the property, that the restriction $\lambda_j : I_{jk} \to \lambda_j(I_{jk})$ is one-to-one for every $k \in \bar{\mathbb{I}}_j$. Let $r_{jk} : \lambda_j(I_{jk}) \to I_{jk}$ denote the inverse mapping. Note that the domain of definition of $r_{jk}$ consists of a closed interval. By the analyticity of $\lambda_j$, $r_{jk}$ is $C^\infty$ in the interior of $\lambda_j(I_{jk})$. Define $K(\lambda)$ by

$$K(\lambda) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \lambda \in \lambda_j(I_{jk})\} \quad \text{for } \lambda \in \mathbb{R}. \quad (2.5)$$

This set is empty, if $\lambda < \lambda_{\min}$. Otherwise, $K(\lambda)$ contains all pairs $(j, k) \in \mathbb{N} \times \mathbb{N}$ having the property, that $\lambda$ is in the domain of definition of $r_{jk}$. According to (A4), $K(\lambda)$ is finite for every $\lambda \geq \lambda_{\min}$. Remember that $\sigma_{\text{res}}(\mathcal{A}) = \{\sigma_1, \sigma_2, \ldots\}$ and note that $K(\lambda)$ is constant on $(\sigma_p, \sigma_{p+1})$ for every $p \in \mathbb{N}$. For every $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$ set

$$K^+(\sigma_p) := \begin{cases} (j, k) \in K(\lambda) \text{ with } \lambda \in (\sigma_p, \sigma_{p+1}) : \frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_p)) = 0 \end{cases},$$

$$K^-(\sigma_p) := \begin{cases} (j, k) \in K(\lambda) \text{ with } \lambda \in (\sigma_{p-1}, \sigma_p) : \frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_p)) = 0 \end{cases} \quad (p \neq 1),$$

$$K^-(\sigma_1) := \emptyset,$$

(2.6)

$$N(j, k, p) := \min\left\{ l \geq 2 : \frac{d^l \lambda_j}{d\xi^l}(r_{jk}(\sigma_p)) \neq 0 \right\} \quad \text{for } (j, k) \in K^+(\sigma_p) \cup K^-(\sigma_p). \quad (2.7)$$

\textbf{Lemma 2.1} 1) If $\sigma_p, \sigma_{p+1} \in \sigma_{\text{res}}(\mathcal{A})$ and $(j, k) \in K(\lambda)$ for $\lambda \in (\sigma_p, \sigma_{p+1})$, then $r_{jk} \in C^\infty(\sigma_p, \sigma_{p+1})$.

2) If $\sigma_p \in \sigma_{\text{res}}(\mathcal{A})$, $p \geq 2$, and if $(j, k) \in K(\sigma_p) \setminus (K^+(\sigma_p) \cup K^-(\sigma_p))$, then $r_{jk} \in C^\infty(\sigma_{p-1}, \sigma_{p+1})$. Furthermore $K(\sigma_1) = K^+(\sigma_1)$. 7
3) If \( \sigma_p \in \sigma_{rs}(A) \) and \((j, k) \in K^+(\sigma_p)\), then there exist constants \(c_{jkpq} \in \mathbb{R}, c_{jkpl} \neq 0 \) and \(\delta > 0\) such that

\[
\frac{dr_{jk}}{d\lambda}(\lambda) = \sum_{q=1}^{\infty} \frac{c_{jkpq}}{|\lambda - \sigma_p|^{-q/N(jk,p)}}
\]

for \(\lambda \in (\sigma_p, \sigma_p + \delta)\). If \((j, k) \in K^-(\sigma_p)\), then (2.8) holds for \(\lambda \in (\sigma_p - \delta, \sigma_p)\) with suitable chosen constants \(c_{jkpq} \in \mathbb{R}, c_{jkpl} \neq 0 \) and \(\delta > 0\).

4) If \(\sigma_p \in \sigma_{rs}(A) \) and \((j, k) \in K^+(\sigma_p) \cup K^-(\sigma_p)\), then either \((j, k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\) or \((j, k-1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\). If \((j, k), (j, k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\), then the coefficients in (2.8) satisfy

\[
c_{jkpl} = (-1)^{N(jk,p)+1} c_{j(k+1)p}.
\]

**Proof:** Note that \(\sigma_{rs}(A) = \bigcup_{j=1}^{\infty} \lambda_j(S_j)\). If \((j, k) \in K(\lambda)\) for \(\lambda \in (\sigma_p, \sigma_{p+1})\), then \((\sigma_p, \sigma_{p+1})\) is subset of the interior of \(\lambda_j(I_{jk})\). This implies \(r_{jk} \in C^\infty(\sigma_p, \sigma_{p+1})\) by the considerations made before the lemma.

If \(\sigma_p \in \sigma_{rs}(A), p \geq 2\), and if \((j, k) \in K(\sigma_p) \setminus (K^+(\sigma_p) \cup K^-(\sigma_p))\), then \((\sigma_{p-1}, \sigma_{p+1})\) is subset of the interior of \(\lambda_j(I_{jk})\) and \(r_{jk} \in C^\infty(\sigma_{p-1}, \sigma_{p+1})\) follows as before. Assume that \((j, k) \in K(\sigma_1)\). Since \(\lambda_j(r_{jk}(\sigma_1)) = \sigma_1 \) and \(\lambda_j(\xi) \geq \sigma_1 = \lambda_{\min}\) for \(\xi \in \mathbb{R}, \frac{d\lambda_j}{d\xi}(r_{jk}(\sigma_1))\) has to vanish. Hence \((j, k) \in K^+(\sigma_1)\). On the other hand, \(K^+(\sigma_1) \subset K(\sigma_1)\) holds by definition.

If \(\sigma_p \in \sigma_{rs}(A) \) and \((j, k) \in K^+(\sigma_p) \cup K^-(\sigma_p)\), then \(r_{jk}(\sigma_p)\) is a boundary point of \(I_{jk}\). This implies that \(\sigma_p = \lambda_j(r_{jk}(\sigma_p))\) is a boundary point of \(\lambda_j(I_{jk})\), the domain of definition of \(r_{jk}\). According to [2], §21, \(r_{jk}\) can be represented by a puiseux-series

\[
r_{jk}(\lambda) = r_{jk}(\sigma_p) + \sum_{q=1}^{\infty} \tilde{c}_{jkpq} |\lambda - \sigma_p|^{q/N(jk,p)}
\]

for \(\lambda \in \lambda_j(I_{jk}) \cap (\sigma_p - \delta, \sigma_p + \delta)\) with suitable chosen \(\tilde{c}_{jkpq} \in \mathbb{R}\) and \(\delta > 0\). Hence (2.8) holds. Since \(\sigma_p = \lambda_j(r_{jk}(\sigma_p))\) is a boundary point of \(\lambda_j(I_{jk})\), \(\sigma_p\) is either a boundary point of \(\lambda_j(I_{jk-1})\) or of \(\lambda_j(I_{jk+1})\). This implies that either \((j, k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\) or \((j, k-1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\). Suppose that \((j, k), (j, k+1) \in K^+(\sigma_p) \cup K^-(\sigma_p)\). Then \(r_{jk}(\sigma_p) = r_{j(k+1)}(\sigma_p)\) and
\(N(j, k, p) = N(j, k + 1, p)\). Relation (2.9) follows inserting the puiseux-series representation of \(r_{jk}\) and \(r_{jk(k+1)}\) into the Taylor expansion of \(\lambda_j\) at \(\xi = r_{jk}(\sigma_p)\) and using that \(\lambda_j(r_{jk}(\lambda)) = \lambda\) for \(\lambda \in \lambda_j(I_{jk}) \cup \lambda_j(I_{jk(k+1)})\). \(\square\)

For \(s \in \mathbb{R}\) define the Banach space \(\mathcal{H}_s\) by

\[
\|\varphi\|_s := \left( \int_{\mathbb{R}} \left(1 + |x|^2\right)^s \|\varphi(x)\|^2_{\mathcal{H}} \, dx \right)^{1/2},
\]

\[
\mathcal{H}_s := L_{2s}(\mathbb{R}, H) := \{ \varphi \in L_{2s,\text{loc}}(\mathbb{R}, H) : \|\varphi\|_s < \infty \} \tag{2.10}
\]

**Lemma 2.2** Suppose that \(s, s' > \frac{1}{2}\) and \(f \in \mathcal{H}_s\). Set

\[
\psi_j(\xi)(x) := e^{i\xi x} \langle (F f)(\xi), v_j(\xi) \rangle_H v_j(\xi) \quad \text{for } \xi, x \in \mathbb{R}, \ j \in \mathbb{N} \tag{2.11}
\]

Then \(\psi_j(\xi) \in \mathcal{H}_{-s'}\) for \(\xi \in \mathbb{R}\) and \(\psi_j \in C^{k,\alpha}(\mathbb{R}, \mathcal{H}_{-s'})\) with \(k \in \mathbb{N}_0\), \(k < \min\{s, s'\} - \frac{1}{2}\) and \(\alpha := \min\{s - k - \frac{1}{2}, s' - k - \frac{1}{2}, 1\}\).

**Proof:** If \(v \in H\), then

\[
\langle (\mathcal{F} f)(\xi), v \rangle_H = F \langle (f(\cdot), v) \rangle_H(\xi)
\]

(see (1.8)). The mapping \(\xi \mapsto F\langle (f(\cdot), v) \rangle_H(\xi)\) is element of \(H^s(\mathbb{R})\), since \(f \in \mathcal{H}_s\). By Sobolev’s Lemma, \(H^s(\mathbb{R}) \subset C^{k,\alpha}(\mathbb{R})\). Since \(v_j\) depends analytically on \(\xi\) by Assumption (A3), the mapping \(\xi \mapsto \langle (\mathcal{F} f)(\xi), v_j(\xi) \rangle_H v_j(\xi)\) is element of \(C^{k,\alpha}(\mathbb{R}, H)\).

Note that \(\mathcal{H}_{-s'}\) is adjoint to \(\mathcal{H}_{s'}\) with respect to the inner product in \(\mathcal{H}\). If \(g \in \mathcal{H}_{s'}\), then

\[
\langle \psi_j(\xi), g \rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{i\xi x} \langle (\mathcal{F} f)(\xi), v_j(\xi) \rangle_H \langle v_j(\xi), g(x) \rangle_H \, dx
\]

\[
= \sqrt{2\pi} \langle (\mathcal{F} f)(\xi), v_j(\xi) \rangle_H \langle v_j(\xi), (\mathcal{F} g)(\xi) \rangle_H.
\]

Hence \(\psi_j \in C^{k,\alpha}(\mathbb{R}, \mathcal{H}_{-s'})\) follows from the first part of the proof. \(\square\)

**Theorem 2.3** Let \(\mathcal{H}\) be a Hilbert-space having the representation (1.7). Suppose that the self-adjoint operator \(A\) is given by (1.9) and that the associated operator family satisfies Assumptions (A1) – (A4) (see page 4). Denote by \(\{P_\lambda\}\) the (left-hand continuous) spectral family of \(A\). Then:
1) The spectrum $\sigma(A)$ of $A$ is purely absolutely continuous and $\sigma(A) = [\lambda_{\text{min}}, \infty)$, where $\lambda_{\text{min}}$ is defined by (2.1). In particular, $A$ has no eigenvalues and $P_\lambda = 0$ for $\lambda \leq \lambda_{\text{min}}$.

2) Let $s, s' > \frac{1}{2}$ and $f \in \mathcal{H}_s$ be fixed (for the definition of $\mathcal{H}_s$ see (2.10)). Then

$$P_\lambda f = \frac{1}{\sqrt{2\pi}} \int_{\lambda_{\text{min}}}^\lambda \sum_{(j,k) \in K(\mu)} \left| \frac{dr_{jk}(\mu)}{d\mu} \psi_j(r_{jk}(\mu)) \right| d\mu$$

(2.12)

for $\lambda > \lambda_{\text{min}}$. (For the definition of $\psi_j$, $K(\mu)$ and $r_{jk}$ see respectively (2.11), (2.5) and the text above (2.5)). Furthermore:

(a) Consider the mapping $P_\lambda f : \mathbb{R} \to \mathcal{H}_{-s'} : \lambda \mapsto P_\lambda f$. For every $\sigma_p, \sigma_{p+1} \in \sigma_{\text{res}}(A)$ (see (2.3)),

$$P_\lambda f \in C^{1,\alpha}((\sigma_p, \sigma_{p+1}), \mathcal{H}_{-s'})$$

with $\alpha := \min\{s - \frac{1}{2}, s' - \frac{1}{2}, 1\}$. 

(2.13)

(b) If $\sigma_p \in \sigma_{\text{res}}(A)$, set

$$N_p := \max \left\{ N(j,k,p) : (j,k) \in K^+(\sigma_p) \cup K^-(\sigma_p) \right\}$$

(2.14)

and suppose that $s, s' > N_p - \frac{1}{2}$ and $f \in \mathcal{H}_s$ are fixed. Then there exist bounded operators $Q_{j,k,p} : \mathcal{H}_s \to \mathcal{H}_{-s'}$, such that

$$\left\| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{(j,k) \in K^+(\sigma_p)} \sum_{l=1}^{N(j,k,p)} \frac{1}{|\lambda - \sigma_{p}|^{1/l(N(j,k,p))}} Q_{j,k,p}(f) \right\|_{-s'} = O\left(|\lambda - \sigma_{p}|^{\alpha/N}\right)$$

(2.15)

as $\lambda \downarrow \sigma_p$ with $\alpha := \min\{s - N_p + \frac{1}{2}, s' - N_p + \frac{1}{2}, 1\}$. The same estimate holds as $\lambda \uparrow \sigma_p$, if $K^+(\sigma_p)$ is replaced by $K^-(\sigma_p)$. Furthermore,

$$Q_{j,k,p}(f)(x) = \frac{1}{\sqrt{2\pi}} |c_{j,k,p}| e^{i x r_{jk}(\sigma_p)} \langle (\mathcal{F} f)(r_{jk}(\sigma_p)), v_j(r_{jk}(\sigma_p)) \rangle_H v_j(r_{jk}(\sigma_p))$$

(2.16)

for $x \in \mathbb{R}$, where $c_{j,k,p}$ denotes the constant of the first summand in (2.8).
Proof: Set

\[ \hat{\mathcal{A}} := \int_{\mathbb{R}} A(\xi) \, d\xi. \]

According to Theorem XIII.85 in [14], \( \hat{\mathcal{A}} \) is self-adjoint in \( \mathcal{H} \). Denote the (left-hand continuous) spectral family in of \( \hat{\mathcal{A}} \) by \( \{ \hat{P}_\lambda \}_{\lambda \in \mathbb{R}} \). Consider a fixed \( \lambda \in \mathbb{R} \) and set \( G(t) := 1 \) if \( t \leq \lambda \), \( G(t) := 0 \) if \( t > \lambda \). Theorem XIII.85 in [14] shows that

\[ \hat{P}_\lambda = G(\hat{\mathcal{A}}) = \int_{\mathbb{R}} G(A(\xi)) \, d\xi = \int_{\mathbb{R}} P^{(\xi)} \, d\xi \]

with \( \{ P^{(\xi)} \} \) denoting the spectral family of \( A(\xi) \) for \( \xi \in \mathbb{R} \). If \( g \in \mathcal{H} \), then

\[ \left( \hat{P}_\lambda g \right)(\xi) = P^{(\xi)} g(\xi) = \sum_{j \in \{ j \in \mathbb{R} : \lambda_j(\xi) < \lambda \}} \langle g(\xi), v_j(\xi) \rangle_H v_j(\xi) \]

for every \( \xi \in \mathbb{R} \) according to Assumption (A2). Note that \( \mathcal{A} = \mathcal{F}^{-1} \hat{\mathcal{A}} \mathcal{F} \) implies that \( P_\lambda = \mathcal{F}^{-1} \hat{P}_\lambda \mathcal{F} \) for \( \lambda \in \mathbb{R} \). Fix \( f \in \mathcal{H}_s \), where \( s > 1/2 \). Then

\[ (\mathcal{F} P_\lambda f)(\xi) = \left( \hat{P}_\lambda \mathcal{F} f \right)(\xi) = \sum_{j \in \{ j \in \mathbb{R} : \lambda_j(\xi) < \lambda \}} \langle (\mathcal{F} f)(\xi), v_j(\xi) \rangle_H v_j(\xi). \]

The right-hand side consists of a finite sum, has bounded support with respect to \( \xi \) (see (A4)) and is a continuous mapping from \( \mathbb{R} \) into \( H \) (see the proof of Lemma 2.2). Set \( L(\lambda) := \{ j \in \mathbb{N} : \lambda_j(\xi) < \lambda \text{ for at least one } \xi \in \mathbb{R} \} \). By (A4), \( L(\lambda) \) is empty or finite for every \( \lambda \in \mathbb{R} \). Application of \( \mathcal{F}^{-1} \) onto the last equation yields that

\[ P_\lambda f = \frac{1}{\sqrt{2\pi}} \sum_{j \in L(\lambda)} \int_{\{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \}} \psi_j(\xi) \, d\xi, \]

where \( \psi_j \) depends continuously on \( \xi \) (see Lemma 2.2). Obviously \( P_\lambda f = 0 \) if \( \lambda \leq \lambda_{\text{min}} \). If \( \lambda > \lambda_{\text{min}} \) and \( j \in L(\lambda) \),

\[ \{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \} = \bigcup_{k \in I_j} (I_{jk} \cap \{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \}) \]

with \( I_{jk} \) being defined by (2.4). Note that the sets on the right-hand side have disjoint interiors. Furthermore, \( I_{jk} \cap \{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \} \neq \emptyset \) only for finitely many...
values of $k$. Substitution $\xi = r_{jk}(\mu)$ yields that

$$
\int_{I_{jk} \cap \{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \}} \psi_j(\xi) \, d\xi = \left\{ 
\begin{array}{ll}
\int_{\lambda_j(I_{jk}) \cap [\lambda_{\text{min}}, \lambda]} \psi_j(r_{jk}(\mu)) \left| \frac{dr_{jk}(\mu)}{d\mu} \right| \, d\mu & \text{if } (j, k) \in \mathcal{K}(\lambda) := \bigcup_{\mu \leq \lambda} K(\mu), \\
0 & \text{otherwise,}
\end{array}
\right.
$$

since the integrand on the right-hand side is continuous in the interior of $\lambda_j(I_{jk})$ and has integrable singularities at the boundary points of $\lambda_j(I_{jk})$ according to (2.8). This implies that

$$
P_\lambda f = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathcal{L}(\lambda)} \sum_{k \in \mathcal{L}_j} \int_{I_{jk} \cap \{ \xi \in \mathbb{R} : \lambda_j(\xi) \leq \lambda \}} \psi_j(\xi) \, d\xi
$$

$$
= \frac{1}{\sqrt{2\pi}} \sum_{(j, k) \in \mathcal{K}(\lambda)} \lambda_j(I_{jk}) \cap [\lambda_{\text{min}}, \lambda] \psi_j(r_{jk}(\mu)) \left| \frac{dr_{jk}(\mu)}{d\mu} \right| \, d\mu
$$

Now (2.12) follows. From (2.12) together with Lemmata 2.1 and 2.2 one obtains that (2.13) holds.

Representation (2.12) shows that $\sigma(A)$ is purely absolutely continuous. For every $\lambda \in [\lambda_{\text{min}}, \infty) \setminus \sigma_{\text{res}}(A)$, there exists $f \in \mathcal{H}_s$ such that $\frac{dP_\lambda f}{d\lambda}(\lambda) \neq 0$. This together with $P_\lambda = 0$ for $\lambda \leq \lambda_{\text{min}}$ proves that $\sigma(A) = [\lambda_{\text{min}}, \infty)$.

If the assumptions of case 2b) are satisfied, Lemma 2.2 implies that

$$
\left\| \psi_j(\xi) - \sum_{l=0}^{N_p-1} \frac{1}{l!} \frac{d^l \psi_j}{d\xi^l}(r_{jk}(\sigma_p)) (\xi - r_{jk}(\sigma_p))^l \right\|_{-\delta'} = O\left( |\xi - r_{jk}(\sigma_p)|^{N_p-1+\alpha} \right)
$$

as $\xi \to r_{jk}(\sigma_p)$, where

$$
f \mapsto \frac{d^l \psi_j}{d\xi^l}(r_{jk}(\sigma_p)): \mathcal{H}_s \to \mathcal{H}_{-\delta'}
$$

are bounded linear operators according to Lemma 2.2. Inserting (2.8) and the corresponding puiseux-series representation of $r_{jk}$ at $\sigma_p$ in the preceding estimate yields (2.15) and (2.16).
3 Initial-value problems

We study the solution \( u \) of (1.11), where \( \mathcal{A} \) denotes a positive self-adjoint operator in a Hilbert-space \( \mathcal{H} \). The long time behaviour of \( u \) is given by Theorem 3.2 and Corollary 3.3.

Denote by \( \{ P_\lambda \}_{\lambda \in \mathbb{R}} \) the (left-hand continuous) spectral family of \( \mathcal{A} \). Then \( P_\lambda = 0 \) if \( \lambda \leq 0 \). For \( r > 0 \), set

\[
\begin{align*}
D(A^r) & := \{ \varphi \in \mathcal{H} : \int_0^\infty \lambda^{2r} d(||P_\lambda \varphi||^2) < \infty \}, \\
A^r \varphi & := \int_0^\infty \lambda^r d(P_\lambda \varphi) \quad \text{if} \ \varphi \in D(A^r). \quad (3.1)
\end{align*}
\]

**Theorem 3.1** Let \( \mathcal{A} \) be a self-adjoint positive operator in the Hilbert-space \( \mathcal{H} \). Then:

1) There is at most one solution \( u \) of the initial-value problem (1.11).

2) If \( \omega \geq 0 \) and \( u_0 \in D(A^{k/2}) \), \( u_1 \in D(A^{(k-1)/2}) \), \( f \in D(A^{(k-2)/2}) \) for some \( k \in \mathbb{N} \), \( k \geq 2 \), then (1.11) has a solution \( u \). Furthermore,

\[
\begin{align*}
A^{(k-j)/2} u & \in C^j([0,\infty), \mathcal{H}) \quad (j \leq k), \\
\frac{d^j(A^{i/2} u)}{dt^j} & = A^{i/2} \left( \frac{d^i u}{dt^i} \right) \quad (j + i \leq k). \quad (3.2)
\end{align*}
\]

**Proof:** If \( u \) solves (1.11) with vanishing data, then \( \frac{d}{dt} (||u(t)||^2 + \langle \mathcal{A} u(t), u(t) \rangle) = 0 \) for \( t > 0 \). This implies that \( u = 0 \).

By functional calculus, the solution of (1.11) is given by

\[
u(t) = \int_0^\infty \psi_\omega(t, \lambda) d(P_\lambda f) + \int_0^\infty \cos \sqrt{\lambda} t d(P_\lambda u_0) + \int_0^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d(P_\lambda u_1), \quad (3.3)
\]
where in the case $\omega > 0$

$$
\psi_\omega(t, \lambda) = \begin{cases} 
\frac{1}{\lambda - \omega^2} \left( e^{-i\omega t} - \cos \sqrt{\lambda} t + \frac{i\omega}{\sqrt{\lambda}} \sin \sqrt{\lambda} t \right) & \text{if } \lambda \in \mathbb{R} \setminus \{\omega^2, 0\}, \\
\frac{1}{2\omega} \left( t e^{-i\omega t} - \frac{1}{\omega} \sin \omega t \right) & \text{if } \lambda = \omega^2, \\
\frac{1}{\omega^2} \left( 1 - e^{-i\omega t} - i\omega t \right) & \text{if } \lambda = 0,
\end{cases}
$$

(3.4)

and

$$
\psi_0(t, \lambda) = \begin{cases} 
\frac{1}{\lambda} \left( 1 - \cos \sqrt{\lambda} t \right) & \text{if } \lambda \in \mathbb{R} \setminus \{0\}, \\
\frac{t^2}{2} & \text{if } \lambda = 0.
\end{cases}
$$

(3.5)

From this (3.2) follows by standard calculations. \qed

Note that $\psi_\omega(t, \omega^2)$ and $\psi_\omega(t, 0)$ are unbounded as $t \to \infty$. The behaviour of $u$ as $t \to \infty$ depends crucially on the behaviour of $P_\lambda$ as $\lambda \to \omega^2$ and $\lambda \downarrow 0$. We require, that the operator $A$ satisfies the following assumption:

**(P1)** $\mathcal{H}$ is continuously imbedded in some Banach space $\mathcal{B}$ and there is a subset $\tilde{\mathcal{B}} \subset \mathcal{H}$, such that for every fixed $g \in \tilde{\mathcal{B}}$ the mapping $P_\lambda : \mathbb{R} \mapsto \mathcal{B} : \lambda \mapsto P_\lambda g$ is differentiable almost everywhere with derivative $\frac{dP_\lambda g}{d\lambda} \in L_1((0, M), \mathcal{B})$ for every $M > 0$.

If $f \in \tilde{\mathcal{B}}$ and $\varphi \in C([0, M])$, then

$$
\int_0^M \varphi(\lambda) d(P_\lambda f) = \int_0^M \varphi(\lambda) \frac{dP_\lambda f}{d\lambda}(\lambda) d\lambda
$$

(3.6)

($M > 0$), where the right-hand side has to be read as a Bochner-integral, see eg. [16].

We will make use of the following theorem:

**Theorem 3.2** Suppose that $A$ is a positive and self-adjoint operator in the Hilbert space $\mathcal{H}$ and that $A$ obeys condition (P1) given above. Let $\omega \geq 0, f \in \tilde{\mathcal{B}}, u_1 \in D(A^{1/2}) \cap \tilde{\mathcal{B}}, u_2 \in D(A) \cap \tilde{\mathcal{B}}$ be given. Furthermore suppose that the following two assumptions are satisfied:

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\textbf{(P2)} There exist $N_f, N_{u_1} \in \mathbb{N}$ and $f_1^{(0)}, \ldots, f_{[N_f/2]}^{(0)}, (u_1)_{1}^{(0)}, \ldots, (u_{[N_u/2]})^{(0)} \in \mathcal{B}$ (with $[\frac{N}{2}] := \max \{ j \in \mathbb{N}_0 : j \leq \frac{N}{2} \}$), such that for respectively $g := f$ and $g := u_1$

$$
\int_0^1 \frac{1}{\sqrt{\lambda}} \left| \frac{dP_\lambda g}{d\lambda}(\lambda) - \sum_{j=1}^{[N_f/2]} \frac{g_j^{(0)}}{\lambda^{1-j/N_f}} \right| d\lambda < \infty.
$$

\textbf{(P3)} There exist $N \in \mathbb{N}_0$ and $f_1^+, \ldots, f_N^+ \in \mathcal{B}$, such that

$$
\int_0^{\omega^2} \frac{1}{|\lambda - \omega^2|} \left| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{j=1}^{N} \frac{f_j^-}{|\lambda - \omega^2|^{1-j/N}} \right| d\lambda < \infty \quad \text{(if $\omega^2 > 0$)},
$$

$$
\int_{\omega^2}^{\omega^2+1} \frac{1}{\lambda - \omega^2} \left| \frac{dP_\lambda f}{d\lambda}(\lambda) - \sum_{j=1}^{N} \frac{f_j^+}{(\lambda - \omega^2)^{1-j/N}} \right| d\lambda < \infty.
$$

Then the solution $u$ of (1.11) given by (3.3) has the following asymptotic behaviour:

1) If $\omega \neq 0$, then

$$
\lim_{t \to \infty} \| u(t) - e^{-i\omega t} I_1(t) - I_2(t) - e^{-i\omega t} u_\omega \|_\mathcal{B} = 0, \quad (3.7)
$$

where

$$
I_1(t) := \sum_{j=1}^{N-1} t^{1-j/N} D_1(1 - \frac{j}{N})(f_j^+ - f_j^-) + i D_2(1 - \frac{j}{N})(f_j^+ + f_j^-)
$$

$$
+ \ln t \cdot (f_N^+ - f_N^-), \quad (3.8)
$$

$$
D_1(\beta) := \frac{\pi}{2\Gamma(\beta+1) \sin \frac{\beta\pi}{2}} \quad \text{for} \quad 0 < \beta < 2, \quad (3.9)
$$

$$
D_2(\beta) := \frac{\pi}{2\Gamma(\beta+1) \cos \frac{\beta\pi}{2}} \quad \text{for} \quad -1 < \beta < 1, \quad (3.10)
$$

$$
I_2(t) := - \sum_{j=1}^{[N_f/2]} t^{1-2j/N_f} \frac{2i D_2(1 - \frac{2j}{N_f})}{\omega} f_j^{(0)}
$$

$$
+ \sum_{j=1}^{[N_u/2]} t^{1-2j/N_u} 2 D_2(1 - \frac{2j}{N_u})(u_1)_j^{(0)}, \quad (3.11)
$$
\[u_\omega := \lim_{\varepsilon \to 0} \left( \int_{|\lambda-\omega|\geq \varepsilon} \frac{1}{\lambda-\omega^2} d(P_\lambda f) \right. \]
\[- \sum_{j=1}^{N-1} \frac{1}{(1 - \frac{j}{N})} \varepsilon^{-j/N} \left( f_j^+ - f_j^- \right) + \ln \varepsilon \cdot \left( f_N^+ - f_N^- \right) \]
\[+ \frac{i\pi}{2} (f_N^+ + f_N^-) + (C_e - \ln(2\omega))(f_N^+ - f_N^-) \quad (3.12)\]

\((C_e := \text{Euler-Mascheroni's constant, the limit has to be taken in } B).\]

2) If \(\omega = 0\), then
\[\lim_{t \to \infty} ||u(t) - I_3(t) - u_0||_B = 0, \quad (3.13)\]

where
\[I_3(t) := \sum_{j=1}^{N-1} t^{2-2j/N} 2D_1(2 - \frac{2j}{N}) f_j^+ + \ln t \cdot 2f_N^+ \]
\[+ \sum_{j=1}^{[N_0/2]} t^{1-2j/N_0} 2D_2(1 - \frac{j}{N_0}) (u_1)_j^{(0)}, \quad (3.14)\]
\[u_0 := \lim_{\varepsilon \to 0} \left( \int_{\varepsilon}^{\infty} \frac{1}{\lambda} d(P_\lambda f) - \sum_{j=1}^{N-1} \frac{f_j^+}{(1 - \frac{j}{N}) 1-j/N} + \ln \delta \cdot f_N^+ \right) \]
\[+ 2C_e f_N^+ \quad (3.15)\]

(the limit has to be taken with respect to the norm in \(B\)).

**Proof:** Note that
\[D_1(\beta) = \begin{cases} \int_0^\infty \frac{1 - \cos \mu}{\mu^{1+\beta}} d\mu & \text{for } 0 < \beta < 2, \\ \end{cases} \]
\[D_2(\beta) = \begin{cases} \int_0^\infty \frac{\sin \mu}{\mu^{1+\beta}} d\mu & \text{for } -1 < \beta < 1. \end{cases} \quad (3.16)\]

Let \(\varepsilon > 0\) be given. The proof proceeds in several steps.

**Step 1:** Since \(H\) is supposed to be continuously imbedded in \(B\), we can choose
\[ M > \omega^2 + 1, \text{ such that} \]
\[
\left\| \int_M^\infty \psi_\omega(t, \lambda) d(P_\lambda f) + \int_M^\infty \cos \sqrt{\lambda} t d(P_\lambda u_0) + \int_M^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d(P_\lambda u_1) \right\|^2_B \\
\leq c \left\| \int_M^\infty \psi_\omega(t, \lambda) d(P_\lambda f) + \int_M^\infty \cos \sqrt{\lambda} t d(P_\lambda u_0) + \int_M^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d(P_\lambda u_1) \right\|^2_H \\
\leq 3c \left( \int_M^\infty 9 d(||P_\lambda f||^2_H) + \int_M^\infty 1 d(||P_\lambda u_0||^2_H) + \int_M^\infty 1 d(||P_\lambda u_1||^2_H) \right) \\
< \varepsilon^2
\]

and
\[
\left\| \int_M^\infty \frac{1}{\lambda - \omega^2} d(P_\lambda f) \right\|_B < \varepsilon.
\]

**Step 2:** By the Riemann-Lebesgue-Lemma (see e.g., [9]), we have
\[
\left\| \int_0^M \cos \sqrt{\lambda} t d(P_\lambda f) \right\|_B = \left\| \text{Re} \int_0^M e^{i\mu t} \frac{dP_\lambda f}{d\lambda} (\mu^2) 2\mu d\mu \right\|_B < \varepsilon
\]
for \( t > T_1 \). Here and in the following, \( T_1, T_2, \ldots \) denote suitable chosen positive real numbers.

**Step 3:** By (P2), we can choose \( \delta > 0 \), such that
\[
\left\| \int_0^\delta \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \left( \frac{dP_\lambda u_1}{d\lambda}(\lambda) - \sum_{j=1}^{[N_{u_1}/2]} \frac{(u_1)_j^{(0)}}{\lambda^{1-j/N_{u_1}}} \right) \right\|_B \\
\leq \int_0^\delta \frac{1}{\sqrt{\lambda}} \left\| \frac{dP_\lambda u_1}{d\lambda}(\lambda) - \sum_{j=1}^{[N_{u_1}/2]} \frac{(u_1)_j^{(0)}}{\lambda^{1-j/N_{u_1}}} \right\|_B d\lambda \\
< \varepsilon.
\]

Riemann-Lebesgue’s Lemma yields as above
\[
\left\| \int_M^\infty \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d(P_\lambda f) \right\|_B < \varepsilon \quad \text{for} \quad t > T_2.
\]

With (3.16), we obtain for \( j \in \{1, \ldots, \left[ \frac{N_{u_1}}{2} \right] \} \)
\[
\int_0^\delta \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \frac{1}{\lambda^{1-j/N_{u_1}}} d\lambda = 2 t^{1-2j/N_{u_1}} \int_0^\infty \frac{\sin \mu}{\mu^{2-2j/N_{u_1}}} d\mu + O\left( \frac{1}{t} \right) \\
= 2 t^{1-2j/N_{u_1}} D_2(1 - \frac{2j}{N_{u_1}}) + O\left( \frac{1}{t} \right)
\]

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as $t \to \infty$. Combining all estimates of this step, we conclude that
\[
\left\| \int_0^M \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} d(P_\lambda u_1) - \sum_{j=1}^{[N_1/2]} t^{1-2j/N_1} 2D_2(1 - 2j/N_1) (u_j)^{(0)} \right\|_B < 3\varepsilon
\]
for $t > T_3$.

**Step 4:** From here to Step 6, the case $\omega \neq 0$ will be considered. Note that
\[
\psi_\omega(t, \lambda) = e^{-i\omega t} \frac{1 - e^{-i(\sqrt{\lambda} - \omega)t}}{\lambda - \omega^2} \left( \frac{i \sin \sqrt{\lambda} t}{\omega \sqrt{\lambda}} + \frac{i \sin \sqrt{\lambda} t}{\omega (\sqrt{\lambda} + \omega)} \right)
\]
The same argument used in Step 3 shows that
\[
\left\| \int_0^M \left( \frac{i \sin \sqrt{\lambda} t}{\omega \sqrt{\lambda}} + \frac{i \sin \sqrt{\lambda} t}{\omega (\sqrt{\lambda} + \omega)} \right) d(P_\lambda f) + \sum_{j=1}^{[N_f/2]} t^{1-2j/N} \frac{2i D_2(1 - 2j/N)}{\omega} f_j^{(0)} \right\|_B < \varepsilon
\]
for $t > T_4$.

**Step 5:** Assumption (P3) implies, that the limit
\[
\mu_\omega := \lim_{\delta \downarrow 0} \left( \int_0^{\omega^2-\delta} \frac{1}{\lambda - \omega^2} \frac{dP_\lambda f}{d\lambda} (\lambda) d\lambda + \sum_{j=1}^{N-1} \frac{f_j^-}{(1 - 4j/N) \delta^1 - j/N - \ln \delta \cdot f_j^-} \right)
\]
exists in $B$. Choose $\delta > 0$ so small, such that
\[
\left\| \int_0^{\omega^2-\delta} \frac{1}{\lambda - \omega^2} \frac{dP_\lambda f}{d\lambda} (\lambda) d\lambda + \sum_{j=1}^{N-1} \frac{f_j^-}{(1 - 4j/N) \delta^1 - j/N - \ln \delta \cdot f_j^-} - \mu_\omega \right\|_B < \varepsilon
\]
and
\[
\int_{\omega^2-\delta}^{\omega^2} \frac{|1 - e^{-i(\sqrt{\lambda} - \omega)t}|}{|\lambda - \omega^2|} \left\| \frac{dP_\lambda f}{d\lambda} (\lambda) - \sum_{j=1}^{N} \frac{f_j^-}{\lambda - \omega^2} \right\|_B d\lambda < \varepsilon
\]
(see (P3)). Let $\beta \in (0, 1)$ be fixed. Standard calculations (see e.g. [11], (6.2)–(6.9) and the equation following (6.29)) show, that for every $\bar{\varepsilon} > 0$, there exists $\delta(\bar{\varepsilon}) > 0$, such that for every $\delta \in (0, \delta(\bar{\varepsilon}))$ there is an $T_5(\bar{\varepsilon}, \delta) > 0$ with
\[
\left\| \int_0^{\omega^2} \frac{1 - e^{-i(\sqrt{\lambda} - \omega)t}}{|\lambda - \omega^2| \beta \delta^3} d\lambda - \frac{1}{\beta \delta^3} + \frac{t^3}{(2\omega)^3} \int_0^{\infty} \frac{1 - e^{i\mu}}{\mu^{1+\beta}} d\mu \right\| < \bar{\varepsilon}
\]
and
\[ \left| \int_{\omega^2 - \delta}^{\omega^2} \frac{1 - e^{-i(\sqrt{\lambda} - \omega)t}}{\lambda - \omega^2} d\lambda + \ln \delta + \ln t + C_i - \ln(2\omega) - \frac{i\pi}{2} \right| < \varepsilon \]

for \( t > T_5(\varepsilon, \delta) \). By Riemann-Lebesgue,
\[ \left| \int_{0}^{\omega^2 - \delta} \frac{e^{-i(\sqrt{\lambda} - \omega)t}}{\lambda - \omega^2} d(P_\lambda f) \right| < \varepsilon \quad \text{for} \ t > T_6, \]

if \( \delta > 0 \) is fixed. This, together with (3.16), proves
\[ \left| \int_{0}^{\omega^2} \frac{1 - e^{-i(\sqrt{\lambda} - \omega)t}}{\lambda - \omega^2} d(P_\lambda f) - \mu_\omega - \left( C_i - \ln(2\omega) - \frac{i\pi}{2} \right) f_N^- \right| + \sum_{j=1}^{N-1} t^{1-j/N} D_1(1 - \frac{i}{N}) - iD_2(1 - \frac{i}{N}) \frac{f_j^- + \ln t \cdot f_N^-}{(2\omega)^{1-j/N}} \right| < 5\varepsilon \]

for \( t > T_7 \).

**Step 6:** As above,
\[ \left| \int_{\omega^2}^{\omega^2 + \delta} \frac{1 - e^{-i(\sqrt{\lambda} - \omega)t}}{\lambda - \omega^2} d(P_\lambda f) - \mu_\omega^+ - \left( C_i - \ln(2\omega) + \frac{i\pi}{2} \right) f_N^+ \right| - \sum_{j=1}^{N-1} t^{1-j/N} D_1(1 - \frac{i}{N}) + iD_2(1 - \frac{i}{N}) \frac{f_j^+ + \ln t \cdot f_N^+}{(2\omega)^{1-j/N}} \right| < \varepsilon \]

for \( t > T_8 \) with
\[ \mu_\omega^+ := \lim_{\delta \to 0} \left( \int_{\omega^2 + \delta}^{\omega^2} \frac{1}{\lambda - \omega^2} dP_\lambda f(\lambda) d\lambda - \sum_{j=1}^{N-1} \frac{f_j^+}{(1 - \frac{i}{N}) \delta^{1-j/N}} + \ln \delta \cdot f_N^+ \right) \]

is proved. Combining the results of Step 1 to Step 6, we obtain that (3.7) holds.

The case \( \omega = 0 \) is proved in the same way by
\[ \left| \int_{0}^{\delta} \frac{1 - \cos \sqrt{\lambda} t}{\lambda^{1+\beta}} d\lambda - 2\beta \int_{0}^{\infty} \frac{1 - \cos \mu}{\mu^{1+2\beta}} d\mu + \frac{1}{\beta \delta^\beta} \right| < \varepsilon \quad (\beta \in (0, 1)), \]
\[ \left| \int_{0}^{\delta} \frac{1 - \cos \sqrt{\lambda} t}{\lambda} d\lambda - 2 \ln t - \ln \delta - 2C_i \right| < \varepsilon \]

for \( t > T_9(\varepsilon, \delta) \) (use (3.67) in [15]). \[ \square \]
Corollary 3.3 (Regularity estimates) Let all Assumptions of Theorem 3.2 be satisfied. In addition, suppose that \( u_0 \in D(A^k) \), \( u_1 \in D(A^{k-1/2}) \), \( f \in D(A^{k-1}) \) for a fixed \( k \in \mathbb{N} \). Denote by \( u \) the solution of \((1.11)\).

1) **General case**: If \( \omega \neq 0 \), then

\[
\lim_{t \to \infty} \| A^j u(t) - e^{-i\omega t} \omega^{2j} I_1(t) - e^{-i\omega t} u_\omega(j) \|_B = 0 \quad \text{for} \quad j = 1, 2, \ldots, k
\]

(3.17)

with \( I_1(t) \) being defined in Theorem 3.2 and

\[
u_\omega(j) := \lim_{\varepsilon \to 0} \left( \int_{|\lambda - \omega^2| \geq \varepsilon} \frac{\lambda^j}{\lambda - \omega^2} d(P_\lambda f) - \sum_{l=1}^{N-1} \left( 1 - \frac{1}{N} \right) \varepsilon^{1-l/N} (f_l^+ + f_l^-) + \ln \varepsilon \cdot \omega^{2j} (f_N^- - f_N^+) \right) + \frac{i\pi}{2} \omega^{2j} (f_N^- + f_N^+) + \omega^{2j} (C_e - \ln(2\omega)) (f_N^- - f_N^+)
\]

(3.18)

\((j \leq k)\), where the limit has to be taken in \( B \). If \( \omega = 0 \), then

\[
\lim_{t \to \infty} \| A^j u(t) - A^{j-1} f \|_B = 0 \quad \text{for} \quad j = 1, 2, \ldots, k.
\]

(3.19)

2) **Principle of limiting Amplitude**: Suppose that additionally \( N_f = N_{u_1} = 0 \) in (P2) of Theorem 3.2 and that \( N = 1 \), \( f_1^+ = f_1^- (= \frac{dP_\lambda f}{d\lambda}(\omega^2)) \) in (P3). Then

\[
\lim_{t \to \infty} \| A^j u(t) - e^{-i\omega t} u_\omega(j) \|_B = 0 \quad \text{for} \quad j = 0, 1, \ldots, k
\]

(3.20)

with

\[
u_\omega(j) := \lim_{\varepsilon \to 0} \int_{|\lambda - \omega^2| \geq \varepsilon} \frac{\lambda^j}{\lambda - \omega^2} d(P_\lambda f) + i\pi \omega^{2j} f_1^+
\]

(3.21)

\((j \leq k)\). The limit in (3.21) has to be taken in \( B \).

**Proof**: For the first part, use the arguments of the proof of Theorem 3.2. The additional assumption is needed only in Step 1. From Step 2 on, replace \( \frac{dP_\lambda f}{d\lambda}(\lambda) \) by \( \lambda^k \frac{dP_\lambda f}{d\lambda}(\lambda) \).

The assertion of the second part follows directly from part 1, if \( j \geq 1 \). The case \( j = 0 \) is obtained from Theorem 3.2. Note that \( f_1^+ = f_1^- = 0 \) if \( \omega = 0 \), since \( P_\lambda f = 0 \) for \( \lambda \leq 0 \). \( \Box \)
The following theorem characterizes the limiting amplitude (3.21) by the principle of limiting absorption.

**Theorem 3.4** Suppose that \( \mathcal{A} \) is a positive and self-adjoint operator in the Hilbert space \( \mathcal{H} \) and that \( \mathcal{A} \) obeys the condition (P1) given before Theorem 3.2. Let \( \omega \geq 0 \) and \( f \in \mathcal{B} \) be given and assume that (P3) of Theorem 3.2 is satisfied with \( N = 1 \) and \( f_1^+ = f_1^- \). Then \( u^{(0)}_\omega \) defined by (3.21) exists in \( \mathcal{B} \) and

\[
\lim_{\tau \downarrow 0} \| R_{\omega^2 + i\tau} f - u^{(0)}_\omega \|_\mathcal{B} = 0. \tag{3.22}
\]

**Proof:** Conclude from (P1) and (P3) (with \( N = 1 \) and \( f_1^+ = f_1^- \)), that the limit \( u^{(0)}_\omega \) exists in \( \mathcal{B} \). Recall that

\[
R_{\omega^2 + i\tau} f = \int_0^\infty \frac{1}{\lambda - \omega^2 - i\tau} d\lambda \mathcal{A}_\lambda f.
\]

Let \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \), such that

\[
\left\| \int_{\omega^2 - \delta}^{\omega^2 + \delta} \frac{1}{\lambda - \omega^2 - i\tau} \left( \frac{d\lambda \mathcal{A}_\lambda f}{d\lambda} - f_1^- \right) d\lambda \right\|_\mathcal{B} < \varepsilon
\]

and

\[
\left\| \int_{|\lambda - \omega^2| \geq \delta} \frac{1}{\lambda - \omega^2} d\lambda \mathcal{A}_\lambda f + i\pi f_1^- - u^{(0)}_\omega \right\|_\mathcal{B} < \varepsilon
\]

with \( u^{(0)}_\omega \) being defined by (3.21). There exists \( \tau_0 > 0 \), such that for \( \tau \in (0, \tau_0) \)

\[
\left\| \int_{|\lambda - \omega^2| \geq \delta} \frac{1}{\lambda - \omega^2 - i\tau} d\lambda \mathcal{A}_\lambda f - \int_{|\lambda - \omega^2| \geq \delta} \frac{1}{\lambda - \omega^2} d\lambda \mathcal{A}_\lambda f \right\|_\mathcal{B} < \varepsilon
\]

and

\[
\left\| \int_{\omega^2 - \delta}^{\omega^2 + \delta} \frac{1}{\lambda - \omega^2 - i\tau} f_1^+ d\lambda - i\pi f_1^+ \right\|_\mathcal{B} < \varepsilon.
\]

This proves (3.22). \( \square \)
4 Spectral properties of elastic operator

Let \( \Omega \subset \mathbb{R}^n \) be given by (1.3) and set

\[
D(\mathcal{A}) := \{ \varphi \in \mathring{H}^1(\Omega)^n : (\Delta + c_0 \text{grad div}) \varphi \in L_2(\Omega)^n \},
\]

\[
\mathcal{A} \varphi := -(\Delta + c_0 \text{grad div}) \varphi \quad \text{for} \; \varphi \in D(\mathcal{A}),
\]

(4.1)

where \( c_0 > -1 \) is supposed, the derivatives have to be taken in distributional sense and \( \mathring{H}^1(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) in \( H^1(\Omega) \). Standard calculations show that

\[
\langle \mathcal{A} \varphi, \varphi \rangle \geq \min \{1, 1 + c_0\} \left( ||\varphi||_1^2 - ||\varphi||_0^2 \right) \quad \text{for} \; \varphi \in D(\mathcal{A})
\]

(4.2)

(\( ||\cdot||_k \) := norm in \( H^k(\Omega)^n \)) and that \( \mathcal{A} \) is self-adjoint in \( L_2(\Omega)^n \).

Note that \( L_2(\Omega)^n = L_2(\mathbb{R}, L_2(D))^n \), since \( \Omega \) is given by (1.3). In this section we define an operator family \( \{ A(\xi) \}_{\xi \in \mathbb{R}} \) in \( L_2(D) \), such that \( \mathcal{A} \) can be represented by (1.9). We show that the theory developed in section 2 can be applied to obtain a representation of the spectral family of \( \mathcal{A} \).

Let the variable in \( D \) be denoted by \((x_2, \ldots, x_n)\) and set

\[
\text{grad}_D := (\partial_2, \ldots, \partial_n)^T, \quad \text{div}_D := (\partial_2, \ldots, \partial_n), \quad \Delta_D := \partial_2^2 + \ldots + \partial_n^2.
\]

Consider the dyadic product \( \text{grad}_D \text{div}_D \) defining a \((n - 1) \times (n - 1)\)-matrix. The \( n \times n \)-matrix given by

\[
\begin{pmatrix}
-\xi^2 & i\xi \text{div}_D \\
i\xi \text{grad}_D & \text{grad}_D \text{div}_D
\end{pmatrix}
\]

contains \((-\xi^2, i\xi \partial_2, i\xi \partial_3, \ldots, i\xi \partial_n)\) in the first line and \((-\xi^2, i\xi \partial_2, i\xi \partial_3, \ldots, i\xi \partial_n)^T\) in the first column. E.g., in the case \( n = 3 \),

\[
\begin{pmatrix}
-\xi^2 & i\xi \partial_2 & i\xi \partial_3 \\
i\xi \partial_2 & i\xi \partial_2^2 & i\xi \partial_2 \partial_3 \\
i\xi \partial_3 & i\xi \partial_3 \partial_2 & i\xi \partial_3^2
\end{pmatrix}
\]
Define the operator $A(\xi)$ for $\xi \in \mathbb{R}$ by

$$
D(A(\xi)) := \left\{ \varphi \in \hat{H}^1(D)^n : \left( \Delta_D + c_0 \begin{pmatrix} 0 & 0 \\ 0 & \text{grad}_D \text{div}_D \end{pmatrix} \right) \varphi \in L_2(D)^n \right\},
$$

$$
A(\xi)\varphi := -\left( \Delta_D - \xi^2 + c_0 \begin{pmatrix} -\xi^2 & i\xi \text{grad}_D \\ i\xi \text{grad}_D & \text{grad}_D \text{div}_D \end{pmatrix} \right) \varphi \quad \text{for } \varphi \in D(A(\xi)).
$$

(4.3)

Let $\xi \in \mathbb{R}$ be fixed. As above,

$$
\langle A(\xi)\varphi, \psi \rangle_D \geq \min\{1, 1 + c_0\} \left( \sum_{j=1}^n \|\text{grad}_D \varphi_j\|^2_D + \xi^2 \|\varphi\|^2_D \right) \geq 0
$$

(4.4)

for $\varphi = (\varphi_1, \ldots, \varphi_n) \in D(A(\xi))$ is proved. Furthermore, $A(\xi)$ is self-adjoint in $L_2(D)^n$.

**Lemma 4.1** Suppose that $D$ has the segment property and that $A(\xi)$ is defined by (4.3). Then:

1) For every fixed $\xi \in \mathbb{R}$, $A(\xi)$ has an orthonormal system of eigenfunctions $\{v_j(\xi)\}_{j \in \mathbb{N}}$ being complete in $L_2(D)^n$. Eigenfunctions and associated eigenvalues $\{\lambda_j(\xi)\}_{j \in \mathbb{N}}$ can be chosen in a way, such that for every fixed $j \in \mathbb{N}$ the mappings $\xi \mapsto v_j(\xi)$, $\xi \mapsto \lambda_j(\xi)$ are analytic on $\mathbb{R}$.

2) For every $\xi \in \mathbb{R}$, $\lambda_j(\xi) > 0$ for $j \in \mathbb{N}$ and $\lambda_j(\xi) \to \infty$ as $j \to \infty$.

3) For every $j \in \mathbb{N}$, there exists $k \in \mathbb{N}$, such that $\lambda_j(\xi) = \lambda_k(-\xi)$ for $\xi \in \mathbb{R}$.

**Proof:** Let $\xi \in \mathbb{R}$ be fixed. Using Rellich’s selection theorem, one obtains from (4.4) that $(A(\xi) + \text{Id})^{-1} : L_2(D)^n \to L_2(D)^n$ is compact. Furthermore this operator is symmetric and positive. The theory of compact symmetric operators shows that $(A(\xi) + \text{Id})^{-1}$ has an orthonormal system $\{v_1(\xi), v_2(\xi), \ldots\}$ of eigenfunctions being complete in $L_2(D)^n$. Denote by $\mu_j(\xi)$ the eigenvalue of $(A(\xi) + \text{Id})^{-1}$ associated to $v_j(\xi)$ ($j \in \mathbb{N}$). Then $\mu_j(\xi) \downarrow 0$ as $j \to \infty$. The operator $A(\xi)$ has the same eigenfunctions with associated eigenvalues $\lambda_j(\xi) = \frac{1 - \mu_j(\xi)}{\mu_j(\xi)} \to \infty$ as $j \to \infty$. 

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Now let $\xi \in \mathbb{R}$ vary. In the notation of [10], \( \{A(\xi)\} \) is a self-adjoint holomorphic operator family of type (A) with compact resolvent. According to Theorem 3.9 of Chapter VII there, eigenvalues and eigenfunctions can be chosen in a way, such that they depend analytically on $\xi$. Since the graphs of $\lambda = \lambda_j(\xi)$ may intersect, enumeration has to be changed eventually. Note that this change doesn’t touch the property $\lambda_j(\xi) \to \infty$ as $j \to \infty$ for fixed $\xi \in \mathbb{R}$. Finally, $\lambda_j(\xi) > 0$ is obtained from (4.4) and from $D(A(\xi)) \subset \hat{H}^1(D)^n$.

Fix $j_0 \in \mathbb{N}$ and consider $u_{j_0}(\xi) = (u_{0j_0}(\xi), \ldots, u_{nj_0}(\xi))^T$, where $u_D := (v_{j_02}, \ldots, v_{jn})^T$. From $A(\xi)(u_{j_0}(\xi)) = \lambda_{j_0}(\xi)(u_{j_0}(\xi))$ and the definition of $A(\xi)$,

$$A(-\xi)\begin{pmatrix} -u_1(\xi) \\ u_D(\xi) \end{pmatrix} = \lambda_{j_0}(\xi)\begin{pmatrix} -u_1(\xi) \\ u_D(\xi) \end{pmatrix}$$

follows. In particular, for every $\xi > 0$, there exists $j \in \mathbb{N}$, such that $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$. This implies, that there is at least one $j \in \mathbb{N}$, such that $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$ for more than countable many values of $\xi \in [0, 1]$. Hence $\lambda_j(-\xi) = \lambda_{j_0}(\xi)$ for $\xi \in \mathbb{R}$ by analyticity of $\lambda_j$ and $\lambda_{j_0}$.

\section*{Lemma 4.2}

Suppose that $\partial D$ has the segment property and let $\{\lambda_j(\xi)\}_{j \in \mathbb{N}}$ denote the eigenvalues of $A(\xi)$ given by Lemma 4.1. Then

$$\lambda_{\min} := \min \{\lambda_j(\xi) : j \in \mathbb{N}, \xi \in \mathbb{R}\} > 0. \quad (4.5)$$

Furthermore, for every $\lambda \geq \lambda_{\min}$, equation $\lambda_j(\xi) = \lambda$ admits only finitely many solutions $(j, \xi) \in \mathbb{N} \times \mathbb{R}$.

\textbf{Proof:} It is sufficient to consider $\lambda_j(\xi)$ with $\xi \geq 0$ according to Item 3 of Lemma 4.1.

Note that

$$2\Re \left< A(\xi)v_j(\xi), \frac{d}{d\xi}v_j(\xi) \right>_D = \lambda_j(\xi) \frac{d}{d\xi} ||v_j(\xi)||^2_D = \lambda_j(\xi) \frac{d}{d\xi} 1 = 0 \quad \text{for} \quad \xi \in \mathbb{R}.$$
From this, one obtains with \( v_j(\xi) = \left( \frac{u_j(\xi)}{u_D(\xi)} \right) \) and \( \xi \geq 0 \) that
\[
\frac{d}{d\xi} \lambda_j(\xi) = \frac{d}{d\xi} \left\langle A(\xi)v_j(\xi), v_j(\xi) \right\rangle_D \\
= 2\Re \left\langle A(\xi)v_j(\xi), \frac{d}{d\xi} v_j(\xi) \right\rangle_D + 2\xi \|v_j(\xi)\|^2_D \\
- c_0 \left\langle \left( -2\xi u_1(\xi) + i \text{div}_{D} u_D(\xi) \right)^T \left( \frac{u_1(\xi)}{u_D(\xi)} \right) \right\rangle_D \\
= 2\xi (\|v_j(\xi)\|^2_D + c_0 \|u_1(\xi)\|^2_D) + 2c_0 \Im \left\langle \text{grad}_{D} u_1(\xi), u_D(\xi) \right\rangle_D \\
\geq -|c_0| (\|\text{grad}_{D} u_1(\xi)\|^2_D + \|u_D(\xi)\|^2_D) \\
\geq -|c_0| (\|\text{grad}_{D} u_1(\xi)\|^2_D + 1),
\]
since \( \|u_D(\xi)\|^2_D \leq \|v_j(\xi)\|^2_D = 1 \). Inequality (4.4) implies that
\[
\|\text{grad}_{D} u_1(\xi)\|^2_D \leq \frac{1}{\min\{1, 1 + c_0\}} \left\langle A(\xi)v_j(\xi), v_j(\xi) \right\rangle_D = \frac{\lambda_j(\xi)}{\min\{1, 1 + c_0\}}.
\]
Hence
\[
\frac{d}{d\xi} \lambda_j(\xi) \geq -|c_0| \frac{|c_0|}{\min\{1, 1 + c_0\}} \lambda_j(\xi) = -|c_0| - c_1 \lambda_j(\xi)
\]
follows. Application of Gronwall's Lemma yields
\[
\lambda_j(\xi) \geq \lambda_j(0) e^{-c_1 \xi} - |c_0| \int_0^\xi e^{-c_1 (\xi-s)} \, ds = \left( \lambda_j(0) + \frac{|c_0|}{c_1} \right) e^{-c_1 \xi} - \frac{|c_0|}{c_1} \quad (4.6)
\]
for \( \xi \geq 0 \). (Set \( \varphi(\xi) := e^{c_1 \xi} \lambda_j(\xi) \) and integrate the resulting differential inequality.)

In order to prove Lemma 4.2, it is convenient to search for eigenvalues \( \lambda_j(\xi) \) being smaller than some given \( \lambda \geq 0 \). From (4.4) one obtains that
\[
\lambda_j(\xi) = \left\langle A(\xi)v_j(\xi), v_j(\xi) \right\rangle_D \geq \min\{1, 1 + c_0\} \xi^2 \|v_j(\xi)\|^2_D = \min\{1, 1 + c_0\} \xi^2
\]
and hence
\[
\lambda_j(\xi) > \lambda \quad \text{for } j \in \mathbb{N}, \xi \geq \sqrt{\frac{\lambda}{\min\{1, 1 + c_0\}}} = c_2.
\]
If \( \xi \in [0, c_2] \), then (4.6) implies that
\[
\lambda_j(\xi) \geq \left( \lambda_j(0) + \frac{|c_0|}{c_1} \right) e^{-c_1 \xi} - \frac{|c_0|}{c_1} > \lambda \quad \text{for } j > J_0
\]
with suitable chosen $J_0 \in \mathbb{N}$ since $\lambda_j(0) \to \infty$ as $j \to \infty$. Hence equation $\lambda_j(\xi) = \lambda$ admits only solutions $(j, \xi) \in \mathbb{N} \times \mathbb{R}$ with $j \leq J_0$ and $\xi \in [0, \omega_2]$. By the analyticity of $\lambda_j(\cdot)$, both assertions of Lemma 4.2 follow. \hfill \Box

**Lemma 4.3** Let $\Omega \subset \mathbb{R}^n$ be given by (1.3), where $D$ is supposed to have the segment property. Set $H := L_2(\Omega)^n = L_2(\mathbb{R}, H)$ with $H = L_2(D)^n$ and denote by $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ the Fourier-transform (see (1.8)). Then

$$\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} = \int_\mathbb{R} \Theta A(\xi) d\xi =: \widehat{\mathcal{A}} \quad (4.8)$$

with $\mathcal{A}$ and $A(\xi)$ being defined by respectively (4.1) and (4.3).

**Proof:** For every $\varphi, \psi \in L_2(D)^n$, the mapping $\xi \mapsto \langle \varphi, (A(\xi) + \text{Id})^{-1} \psi \rangle_D$ is continuous on $\mathbb{R}$ by Lemma 4.1, and hence measurable. This shows that $\int_\mathbb{R} \Theta A(\xi) d\xi$ is defined. It is sufficient to prove $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \subset \widehat{\mathcal{A}}$. This implies that $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} = \widehat{\mathcal{A}}$, since both operators are self-adjoint. (For the case of $\widehat{\mathcal{A}}$ see Theorem XIII.85 in [14].) Suppose that $f \in D(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1})$, which means that $\mathcal{F}^{-1}f \in D(\mathcal{A})$. Since $D(\mathcal{A}) \subset \dot{H}^1(\Omega)^n$, there exists a sequence $\{\varphi_j\}$ in $C_0^\infty(\Omega)^n$ with

$$\|\varphi_j - \mathcal{F}^{-1}f\|_{H^1(\Omega)^n} \to 0 \quad \text{as } j \to \infty.$$ 

Since $\mathcal{F}$ is norm-invariant and commutes with derivatives with respect to $x_2, \ldots, x_n$, this implies that

$$\int_\mathbb{R} \|((\mathcal{F} \varphi_j)(\xi) - f(\xi))\|_{H^1(D)^n}^2 d\xi \to 0 \quad \text{as } j \to \infty.$$ 

There exists a subsequence of $\{\varphi_j\}$, again denoted by $\{\varphi_j\}$, such that

$$\|((\mathcal{F} \varphi_j)(\xi) - f(\xi))\|_{H^1(D)^n}^2 \to 0 \quad \text{as } j \to \infty \text{ a.e. on } \mathbb{R}.$$ 

Note that $(\mathcal{F}f)(\xi) \in C_0^\infty(D)^n$ for every $\xi \in \mathbb{R}$. Hence $f(\xi) \in \dot{H}^1(D)^n$ a.e. for $\xi \in \mathbb{R}$.

If $\varphi \in C_0^\infty(\Omega)^n$, then $(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1})\varphi(\xi) = \mathcal{A}(\xi)\varphi(\xi, \cdot)$ and

$$\langle \mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}f, \varphi \rangle_\Omega = \langle f, \mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}\varphi \rangle_\Omega = \int_\mathbb{R} \langle \mathcal{F}(\xi), \mathcal{A}(\xi)\varphi(\xi, \cdot) \rangle_D d\xi$$

$$= \int_\mathbb{R} \left\langle \left( -\Delta_D + \xi^2 - \omega_0 \left( \begin{array}{c} -\xi^2 \\ \text{grad}_D \\ \text{div}_D \end{array} \right) \right) f(\xi), \varphi(\xi, \cdot) \right\rangle_D d\xi$$
by the symmetry of $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1}$, Fubini's theorem and definition of weak derivative. Again using Fubini's theorem, one obtains that

$$\left( -\Delta_D + \xi^2 - c_0 \left( \frac{-\xi^2}{\xi^{\text{grad}_D}} \frac{i\xi^{\text{div}_D}}{\text{grad}_D \text{div}_D} \right) \right) f(\xi) = (\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f)(\xi) \in L_2(D)^n$$

a.e. for $\xi \in \mathbb{R}$, and hence that $f(\xi) \in D(\mathcal{A}(\xi))$ and $A(\xi)f(\xi) = (\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f)(\xi)$ a.e. for $\xi \in \mathbb{R}$. Furthermore,

$$\int_{\mathbb{R}} \|A(\xi)f(\xi)\|^2_D \, d\xi = \int_{\mathbb{R}} \|(\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f)(\xi)\|^2_D \, d\xi < \infty$$

follows. This implies that $f \in D(\hat{\mathcal{A}})$ and

$$(\hat{\mathcal{A}} f)(\xi) = A(\xi)f(\xi) = (\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} f)(\xi) \quad \text{a.e. for } \xi \in \mathbb{R},$$

which concludes the proof of $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \subset \hat{\mathcal{A}}$. \hfill \Box

The following corollary is obtained from Lemmata 4.1, 4.2, 4.3 and from (4.4), (4.7):

**Corollary 4.4** Suppose that $D \subset \mathbb{R}^{n-1}$ is bounded and has the segment property. Then:

1) The operator family $\{A(\xi)\}_{\xi \in \mathbb{R}}$ given by (4.3) satisfies Assumptions (A1) - (A4) (see page 4).

2) All assertions of Theorem 2.3 are valid for spectrum and spectral family of the operator $\mathcal{A}$ given by (4.1).

5 Elastic wave equation

We study the long-time behaviour of $u$ solving (1.2). $L_2$-estimates are given by Theorem 5.1, if $u$ is a strong solution. Pointwise estimates of the classical solution are presented in Theorem 5.2.
In Section 2, Banach-spaces $H_s = L_2^{(s)}(\mathbb{R}, H)$ with associated norm $\|\cdot\|_s$ ($s \in \mathbb{R}$) were defined (see (2.10)). In the following application, $H := L_2(D)^n$ has to be set, since $\Omega = \mathbb{R} \times D$. Note that

$$H_s = H_0^0(\Omega)^n, \quad \|\cdot\|_s \text{ is equivalent to } \|\cdot\|_{0,s},$$

where

$$H_s^k(\Omega)^n := \left\{ \varphi \in H_s^k(\Omega)^n : \|\varphi\|_{k,s} < \infty \right\},$$

$$\|\varphi\|_{k,s} := \left( \int_{\Omega} (1 + |x|^2)^s \left( \sum_{|\alpha| \leq k} |D^\alpha \varphi(x)|^2 \right) \, dx \right)^{1/2}$$

(5.1)

($k \in \mathbb{N}_0$, $s \geq 0$).

**Theorem 5.1** Let $\Omega \subset \mathbb{R}^n$ be given by (1.3) with $D$ having the segment property, and let $A$ be the elastic spatial operator defined by (4.1). Denote by $u$ the solution of (1.11) (with $H := L_2(\Omega)^n$), where

$$\omega \geq 0, \quad u_0 \in D(A), \quad u_1 \in D(A^{1/2}), \quad f \in L_2(\Omega)^n$$

is supposed.

1) **Principle of limiting amplitude:** If, in addition to (5.2), $\omega^2 \notin \sigma_{\text{res}}(A)$ (see (2.2)) and $u_0, u_1, f \in H_s^0(\Omega)^n$ for some $s > \frac{1}{2}$, then

$$u(t) = e^{-\omega t} u_\omega + r(t),$$

(5.3)

where

$$\|r(t)\|_{0,s'} = o(1) \quad \text{as } t \to \infty$$

(5.4)

for every $s' > \frac{1}{2}$ and $u_\omega := u_0^{(0)}$ is given by (3.21) (with $B := H_s^{\omega'}(\Omega)^n$).

2) **Resonance case:** If, in addition to (5.2), $\omega^2 = \sigma_p \in \sigma_{\text{res}}(A)$ and $u_0, u_1, f \in H_s^0(\Omega)^n$ for some $s > N_p + \frac{1}{2}$ (see (2.14)), then resonance of order $t^{1-N_p}$
occurs. In particular,

\[ u(t) = e^{-i\omega t} \sum_{(j,k) \in K^+ (\sigma_p)} \left( \sum_{l=1}^{N(j,k,p)-1} t^{1-l/N(j,k,p)} \times \right. \]
\[ \left. \times \frac{D_1(1 - \frac{l}{N(j,k,p)}) + iD_2(1 - \frac{l}{N(j,k,p)})}{(2\omega)^{1-l/N(j,k,p)}} Q_{j,k,p}(f) \right) + \ln t \cdot Q_{j,k,p}(f)(\tilde{f}) \]
\[ - e^{-i\omega t} \sum_{(j,k) \in K^- (\sigma_p)} \left( \sum_{l=1}^{N(j,k,p)-1} t^{1-l/N(j,k,p)} \times \right. \]
\[ \left. \times \frac{D_1(1 - \frac{l}{N(j,k,p)}) - iD_2(1 - \frac{l}{N(j,k,p)})}{(2\omega)^{1-l/N(j,k,p)}} Q_{j,k,p}(f) \right) + \ln t \cdot Q_{j,k,p}(f)(\tilde{f}) \right) + e^{-i\omega t} u_\omega + r(t), \quad (5.5) \]

where \( r \) satisfies (5.4) for every \( \varepsilon > N_p + \frac{1}{2} \) and where at least one term of order \( t^{1-1/N(j,k,p)} \) with \( N(j,k,p) = N_p \) does not vanish and is not cancelled out by other terms, if \( f \) is chosen suitable. Furthermore, \( u_\omega \) is given by

\[ u_\omega = \lim_{\varepsilon \downarrow 0} \left( \int_{|\lambda - \sigma_p| \geq \varepsilon} \frac{1}{\lambda - \sigma_p} d(P_\lambda f) \right. \]
\[ \left. - \sum_{(j,k) \in K^+ (\sigma_p)} \left( \sum_{l=1}^{N(j,k,p)-1} \frac{1}{1 - \frac{l}{N(j,k,p)}} \varepsilon^{1-l/N(j,k,p)} Q_{j,k,p}(f) \right) \right. \]
\[ \left. + \left( \frac{i\pi}{2} + C_e - \ln(2\omega) \right) Q_{j,k,p}(f)(\tilde{f}) \right) \]
\[ + \sum_{(j,k) \in K^- (\sigma_p)} \left( \sum_{l=1}^{N(j,k,p)-1} \frac{1}{1 - \frac{l}{N(j,k,p)}} \varepsilon^{1-l/N(j,k,p)} Q_{j,k,p}(f) \right) \]
\[ \left. + \left( \frac{i\pi}{2} - C_e + \ln(2\omega) \right) Q_{j,k,p}(f)(\tilde{f}) \right) \right), \quad (5.6) \]

where the limit has to be taken in \( H^0_{\lambda'}(\Omega)^n \) for \( \lambda' > N_p + \frac{1}{2} \).
Proof: According to Corollary 4.4, the spectral family of $A$ and its properties are given by (2.12), (2.13) and (2.15). Application of Corollary 3.3, Part 2 (with $k = 1$ and $\hat{B} := H_{s} = H_{s}^{0}(\Omega)$, $\hat{B} := H_{-s} = H_{-s}^{0}(\Omega)$) yields (5.3) and (5.4). From Item 1 of Theorem 3.2, (5.5) with (5.4) and (5.6) is obtained.

It remains to prove, that resonance of order $t^{1-1/N_{p}}$ occurs, if $f$ is chosen suitable. Note that every term on the right-hand side of (5.5) with $N(j, k; p) < N_{p}$ or with $l > 1$ grows slower than $t^{1-1/N_{p}}$. Thus only summands with $N(j, k; p) = N_{p}$ and $l = 1$ have to be studied. Choose one pair $(j_{0}, k_{0}) \in K^{+}(\sigma_{p}) \cup K^{-}(\sigma_{p})$ such that $N(j_{0}, k_{0}; p) = N_{p}$. Furthermore, choose $f \in H_{0}^{0}(\Omega)$ such that $(\langle F f \rangle(r_{j_{0}k_{0}}(\sigma_{p})), v_{j_{0}}(r_{j_{0}k_{0}}(\sigma_{p}))) \neq 0$ (Note that it is possible to suppose $f \in C_{0}^{\infty}(\Omega)$). Then $Q_{j_{0}k_{0}p}(f) \neq 0$ according to (2.16). The associated resonance term is not cancelled out by the other terms on the right-hand side of (5.5), which will be shown in the following.

Consider another pair $(j, k) \in K^{+}(\sigma_{p}) \cup K^{-}(\sigma_{p})$ with $N(j, k; p) = N_{p}$. If $r_{jk}(\sigma_{p}) \neq r_{j_{0}k_{0}}(\sigma_{p})$, then $Q_{jk}p(f)$ and $Q_{j_{0}k_{0}p}(f)$ are linearly independent due to the factor $e^{i\pi r_{jk}(\sigma_{p})}$ in the definition (2.16) of $Q_{jk}p(f)$. If $r_{jk}(\sigma_{p}) = r_{j_{0}k_{0}}(\sigma_{p})$ and $j \neq j_{0}$, then $Q_{jk}p(f)$ and $Q_{j_{0}k_{0}p}(f)$ are again linearly independent, since $(v_{j}(r_{j_{0}k_{0}}(\sigma_{p})))_{j \in \mathbb{N}}$ is supposed to be an orthonormal system. It remains to consider the case $r_{jk}(\sigma_{p}) = r_{j_{0}k_{0}}(\sigma_{p})$ and $j = j_{0}$. In this case, $k = k_{0} \pm 1$ according to Item 4 of Lemma 2.1 and construction of $r_{jk}$. Furthermore $Q_{jk}p(f) = Q_{j_{0}k_{0}p}(f)$ holds (see (2.9) and (2.16)). If $(j, k, (j_{0}, k_{0}))$ are contained in the same set $K^{+}(\sigma_{p})$ (or respectively $K^{-}(\sigma_{p})$), then both associated summands in (5.5) lead to the same resonance term, so that the factor of $t^{1-1/N_{p}}Q_{j_{0}k_{0}p}(f)$ is multiplied by two. If e.g., $(j, k) \in K^{+}(\sigma_{p})$ and $(j_{0}, k_{0}) \in K^{-}(\sigma_{p})$, then the factor of $t^{1-1/N_{p}}Q_{j_{0}k_{0}p}(f)$ is given by

$$
\frac{D_{1}(1 - \frac{1}{N_{p}}) + i D_{2}(1 - \frac{1}{N_{p}})}{(2\omega)^{1-1/N_{p}}} - \frac{D_{1}(1 - \frac{1}{N_{p}}) - i D_{2}(1 - \frac{1}{N_{p}})}{(2\omega)^{1-1/N_{p}}} = 2i \frac{D_{2}(1 - \frac{1}{N_{p}})}{(2\omega)^{1-1/N_{p}}} \neq 0.
$$

The same happens if $(j, k) \in K^{-}(\sigma_{p})$ and $(j_{0}, k_{0}) \in K^{+}(\sigma_{p})$.  

\[\square\]
Theorem 5.2 Let $\Omega$ and $A$ be given by respectively (1.3) and (4.1). Suppose that 
$\omega \geq 0$ and that 
\[
\partial D \in C^K, \quad u_0 \in D(A^{K'}) \quad u_1 \in D(A^{K'-1/2}) \quad f \in D(A^{K'-1}) \tag{5.7}
\]
for some $K > \frac{r}{2} + 2$, $K' > \frac{n}{4} + 1$. Then Problem (1.2) has a solution 
$u \in C^2([0, \infty) \times \Omega)$. It is the only solution of (1.2) satisfying 
\[
u \in C^2([0, \infty), L_2(\Omega)), \quad u(t) \in D(A) \text{ for } t \geq 0. \tag{5.8}
\]
Furthermore the following asymptotic estimates hold:

1) **Principle of limiting amplitude:** If, in addition to (5.7), $\omega^2 \not\in \sigma_{\text{res}}(A)$ and 
$u_0, u_1, f \in H^s_0(\Omega)^n$ for some $s > \frac{1}{2}$, then 
\[
u(t, x) = e^{-\omega t} u_\omega(x) + r(t, x), \tag{5.9}
\]
where $u_\omega \in C^2(\overline{\Omega})$ is solution of (1.6) and, for every $s' > \frac{1}{2}$, 
\[
\frac{1}{(1 + |x|^2)^{s'/2}} r(t, x) = o(1) \quad \text{as } t \to \infty, \text{ uniformly with respect to } x \in \overline{\Omega}. \tag{5.10}
\]

2) **Resonance case:** If, in addition to (5.8), $\omega^2 \in \sigma_{\text{res}}(A)$ and $u_0, u_1, f \in H^s_0(\Omega)^n$ 
for some $s > N_p + \frac{1}{2}$, then $u$ satisfies (5.5) pointwise with respect to $x \in \overline{\Omega}$, 
where $r(t, x)$ obeys (5.10) for every $s' > N_p + \frac{1}{2}$ and $u_\omega \in C^2(\overline{\Omega})$ solves (1.6).

For the proof, the following lemma is needed. It can be found in [7], but without proof. For the idea of the proof see e.g. Hilfssatz 1.15 in [12].

**Lemma 5.3** Set $D := -(\Delta + c_0 \text{ grad div})$ (in distributional sense) and 
$\mathcal{H}^s_0(\Omega)^n :=$ completion of $C^s_0(\Omega)^n$ in $H^s_0(\Omega)^n$ ($s \geq 0$). Suppose that $\Omega$ is given by (1.3) with 
$\partial D \in C^K$ for some $K \geq 2$ and that $\varphi \in H^s_0(\Omega)^n$ satisfies 
\[
D^j \varphi \in \mathcal{H}^s_0(\Omega)^n \quad \text{for } j = 0, 1, \ldots, K' - 1, \quad D^K \varphi \in H^s_0(\Omega)^n
\]
for some $K' \in \mathbb{N}$ and some $s \geq 0$. Then $\varphi \in H^s_0(\Omega)^n$ and 
\[
\|\varphi\|_{H^s_0(\Omega)^n} \leq c(K, K', s, \Omega) \left(\|D^K \varphi\|_{H^s_0(\Omega)^n} + \|\varphi\|_{H^s_0(\Omega)^n}\right) \tag{5.11}
\]
with $c(K, K', s, \Omega) > 0$ not depending on $\varphi$. 

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Proof of Theorem 5.2: Denote by \( u \) the solution of (1.11) given by (3.3). From (5.7) and Theorem 3.1, one obtains that \( \mathcal{D}^{K'-j} \frac{\partial u}{\partial x_j} \in C([0, \infty), L^2(\Omega)^n), j \leq K' \). Application of Lemma 5.3 (with \( s = 0 \)) and Sobolev’s imbedding theorem yields \( u \in C^2([0, \infty) \times \bar{\Omega}) \). By definition of \( \mathcal{A} \) (see (4.1)), \( u \) solves (1.2). Uniqueness of the solution of (1.2) satisfying (5.8) follows from the uniqueness of the solution of (1.11) asserted by Theorem 3.1.

Suppose that all assumptions of Part 2 are satisfied and that \( s' > N_p + \frac{1}{2} \). Note that \( \omega^2 = \sigma_p > 0 \) by (4.5) and (2.3). According to (5.5) and Corollary 3.3, there exist constants \( N \in \mathbb{N}, \alpha_1, \ldots, \alpha_N \in [0, 1), \alpha_1 < \ldots < \alpha_N \) and operators \( Q_0, \ldots, Q_N \), such that

\[
\left\| \mathcal{D}^j u(t) - e^{-i\omega t} \omega^2 j \left( \sum_{k=1}^{N} i^{\alpha_k} Q_k(f) + \ln t \cdot Q_0(f) \right) - e^{-i\omega t} u_0^{(j)} \right\|_{0,-s'} \rightarrow 0
\]

as \( t \rightarrow \infty \) \( (j = 0, 1, \ldots, K') \). Set

\[
v(t) := t^{-\alpha_N} e^{i\omega t} u(t) \quad \text{for } t \geq 0.
\]

Then \( v(t), \mathcal{D}v(t), \ldots, \mathcal{D}^{K'} v(t) \) converge as \( t \rightarrow \infty \) with respect to \( \| \cdot \|_{0,-s'} \). Furthermore \( v(t), \mathcal{D}v(t), \ldots, \mathcal{D}^{K'-1} v(t) \in \tilde{H}^1_{-s'}(\Omega)^n \) for every \( t \geq 0 \) (since \( v(t) \in D(\mathcal{A}^{K'}) \)). Application of Lemma 5.3 yields

\[
\|v(t) - v(t')\|_{\min(K,2K'),-s'} \rightarrow 0 \quad \text{as } t, t' \rightarrow \infty.
\]

This implies that \( Q_N(f) = \lim_{t \rightarrow \infty} v(t) \in H^1_{-s'}(\Omega)^n, \mathcal{D}^j Q_N(f) = \omega^2 j Q_N(f) \) \( (j = 1, \ldots, k) \) and \( \mathcal{D}^j Q_N(f) \in \tilde{H}^1_{-s'}(\Omega)^n \) for \( j = 0, 1, \ldots, k-1 \). Repeat the same argument with \( v(t) := t^{-\alpha_{N-1}} e^{i\omega t} (u(t) - t^{\alpha_N} Q_N(f)) \) and so on. This proves that

\[
\|v(t)\|_{\min(K,2K'),-s'} \rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]
Now use Sobolev’s inequality to obtain
\[
\left| \frac{r(t,x)}{(1+|x|^2)^{s/2}} \right| \leq c_1 \left\| \frac{r(t)}{(1+|x|^2)^{s/2}} \right\|_{\min(K,2K')}^{0,0} \\
\leq c_2 \|r(t)\|_{\min(K,2K'),-s'} \\
\to 0 \quad \text{as } t \to \infty
\]
uniformly with respect to \( x \in \Omega \). Thus (5.10) is proved in the resonance case. Finally note that \( u_\omega^{(0)} \in H^{\min(K,2K')}_{-s'} \subset C^2(\Omega) \) and
\[
\langle (\mathcal{D} - \omega^2)u_\omega^{(0)}, \varphi \rangle = \langle u_\omega^{(0)}, (\mathcal{D} - \omega^2)\varphi \rangle \\
= \lim_{t \to \infty} \left\langle e^{i\mathcal{D}t} u(t) - \sum_{k=1}^{N} e^{i\mathcal{D}t} Q_k(f) - \ln t \cdot Q_0(f), (\mathcal{D} - \omega^2)\varphi \right\rangle \\
= \lim_{t \to \infty} \langle (\mathcal{D} - \omega^2)e^{i\mathcal{D}t} u(t), \varphi \rangle \\
= \langle f, \varphi \rangle
\]
for every \( \varphi \in C^\infty_0(\Omega)^n \), since \( (\mathcal{D} - \omega^2)Q_k(f) = 0 \) and
\[
\left\| e^{i\mathcal{D}t} (\mathcal{D} - \omega^2)u(t) - f \right\|_{0,-s'} = \left\| e^{i\mathcal{D}t} (A - \omega^2)u(t) - f \right\|_{0,-s'} \to 0 \quad \text{as } t \to \infty,
\]
which follows by the same arguments used in the proof of Theorem 3.2. This shows that \( u_\omega = u_\omega^{(0)} \) solves (1.6). Part 1 of the theorem is proved in the same way. \( \square \)

References


