Recent Results on Lieb-Thirring Inequalities

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We give a survey of results on Lieb-Thirring inequalities for the eigenvalue moments of Schrödinger operators. In particular, we discuss the optimal values of the constants therein for higher dimensions. We elaborate on certain generalisations and some open problems as well.

0. Introduction

1. Let H be the Schrödinger operator

$$H(V;\hbar) = -\hbar^2\Delta - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d).$$

For suitable real-valued potential wells V the negative spectrum $\{\lambda_n(V;\hbar)\}$ of H is semi-bounded from below and discrete.

2. For $\sigma \geq 0$ let

$$S_{\sigma,d}(V;\hbar) = \operatorname{tr} H_-^\sigma(V;\hbar) = \sum_n (-\lambda_n(V;\hbar))^\sigma$$

be the σ-Riesz mean of the negative spectrum. Moreover, let

$$S^{\mathrm{cl}}_{\sigma,d}(V;\hbar) = \int\int_{h<0} (-h(\xi,x))^\sigma \frac{dx\,d\xi}{(2\pi\hbar)^d}$$

be the σ-means of the symbol $h = |\xi|^2 - V(x)$.

For appropriate pairs of σ and d the Lieb-Thirring inequalities states that

$$S_{\sigma,d}(V;\hbar) \quad \leq \quad R(\sigma,d) S^{\text{cl}}_{\sigma,d}(V;\hbar)$$

3. The Lieb-Thirring inequality captures the correct order of the semi-classical Weyl type asymptotics

$$S_{\sigma,d}(V;\hbar) = (1 + o(1)) S^{\text{cl}}_{\sigma,d}(V;\hbar) \quad \text{as} \quad \hbar \to 0.$$

The inequality holds for *all* positive values of $\hbar$.

It extracts hard information on the negative spectrum of Schrödinger operators from the classical systems in the *non-asymptotical* regime.

4. The ξ-integration in $S^{\mathsf{cl}}_{\sigma,d}$ evaluates to

$$S^{\mathsf{cl}}_{\sigma,d}(V;\hbar) = L^{\mathsf{cl}}_{\sigma,d}\hbar^{-d}\int V_+^{\sigma+\frac{d}{2}}dx,$$

where

$$L^{\mathsf{cl}}_{\sigma,d} = \frac{\Gamma(\sigma+1)}{2^d\pi^{d/2}\Gamma\left(\sigma+\frac{d}{2}+1\right)}.$$

The Lieb-Thirring inequality turns into

$$\sum_n (-\lambda_n(V;\hbar))^\sigma \leq L_{\sigma,d}\hbar^{-d}\int V_+^{\sigma+\frac{d}{2}}dx$$

with the usual Lieb-Thirring constants

$$L_{\sigma,d} = R(\sigma,d)L^{\mathsf{cl}}_{\sigma,d}.$$

In view of the Weyl asymptotics we have

$$R(\sigma, d) \geq 1 \quad \text{and} \quad L_{\sigma,d} \geq L^{\mathrm{cl}}_{\sigma,d}.$$

5. One should ask the following questions:

1. For which σ and d does the inequality

$$S_{\sigma,d}(V; \hbar) \leq R(\sigma, d) S^{\mathrm{cl}}_{\sigma,d}(V; \hbar)$$

actually hold?

2. What are the sharp values of $R(\sigma, d)$?

3. For which σ and d is $R(\sigma, d) = 1$?

1. Counterexamples:

For $d = 1, 2$ any arbitrary small attractive potential well will couple at least one bound state. Hence, we have $S_{0,d}(V;\hbar) \geq 1$ while $S^{\mathrm{cl}}_{0,d}(V;\hbar) \sim \int V^{d/2} dx$ can be arbitrary small.

This contradicts to LTH for $\sigma = 0$ and $d = 1, 2$.

For $d = 1$ the weakly coupled bound state satisfies

$$\lambda_1(V;\hbar) = -\frac{1}{4\hbar^2}\left(\int V dx\right)^2 + o(\hbar^{-2}), \quad \hbar \to \infty.$$

This implies $S_{\sigma,1}(V;\hbar) = O(\hbar^{-2\sigma})$ while $S^{\mathrm{cl}}_{\sigma,1}(V;\hbar) = O(\hbar^{-1})$ for $\hbar \to \infty$.

This excludes LTH for $d = 1$ and $0 < \sigma < 1/2$.

2. The LTH inequality holds true for

$$\begin{array}{lll} \sigma \geq 1/2 & \text{if} & d = 1 \\ \sigma > 0 & \text{if} & d = 2 \\ \sigma \geq 0 & \text{if} & d \geq 3 \end{array} .$$

[LTh] for $\sigma > 1/2,\ d = 1$ and $\sigma > 0,\ d \geq 2$; [C,L,R] for $\sigma = 0,\ d \geq 3$; [W] for $\sigma = 1/2, d = 1$.

The parameter $\hbar$ can be scaled out and we put $\hbar = 1$.

3. Borderline cases are the most complicated ones. In particular, for $\sigma = 0$ and $d \geq 3$ LTH turns into the celebrated CLR estimate on the number of bound states

$$\text{rank}\, H_-(V) = S_{0,d}(V) \leq L_{0,d} \int V_+^{d/2} dx.$$

CLR implies LTH for $\sigma > 0$ and $d \geq 3$. Indeed,

$$\begin{aligned} S_{\sigma,d}(V) &= \sum_n (-\lambda_n(V))^\sigma \\ &= \frac{1}{\sigma}\int_0^\infty dt t^{\sigma-1} \underbrace{S_{0,d}(V-t)}_{\#\{\lambda_n<-t\}} \\ &\leq \frac{R(0,d)}{\sigma}\int_0^\infty dt t^{\sigma-1} \underbrace{S^{\mathrm{cl}}_{0,d}(V-t)}_{\frac{\mathrm{vol}\,\{(\xi,x):h<-t\}}{(2\pi)^d}} \\ &\leq R(0,d) S^{\mathrm{cl}}_{\sigma,d}(V). \end{aligned}$$

In a similar way one shows that $R(\sigma',d) \leq R(\sigma,d)$ for all $\sigma' \geq \sigma$ [Aizenman,Lieb].

4. In the other borderline case $\sigma = 1/2$ and $d = 1$ for $V \geq 0$ one finds in fact a *two-sided* estimate

$$S^{\mathrm{cl}}_{1,\frac{1}{2}}(V) \leq S_{1,\frac{1}{2}}(V) \leq 2S^{\mathrm{cl}}_{1,\frac{1}{2}}(V)$$

[GGM], [W], [HLT]. Note that $\sigma = 1/2$ and $d = 1$ is the only point in the Lieb-Thirring scale, where such a two-sided estimate is possible.

2. On the sharp values of $R(\sigma, d)$.

1. The dimension $d = 1$: Sharp constants appear already in [LTh], [AL]

$$R(\sigma, 1) = 1 \quad \text{for all} \quad \sigma \geq 3/2.$$

It uses a *trace identity* for $\sigma = 3/2$ and the monotonicity argument [AL].
The only other case settled was $\sigma = 1/2$ with

$$R(1/2, 1) = 2$$

in [HLT]. This reflects the weak coupling behaviour.

The optimal values of $R(\sigma, 1)$ for $1/2 < \sigma < 3/2$ are unknown. An analysis of the lowest bound state shows that here

$$R(\sigma, 1) \geq \sup_{V \in L^{\sigma+\frac{1}{2}}} \frac{(-\lambda_1(V))^\sigma}{S^{\mathrm{cl}}_{\sigma,1}(V)} = 2 \left(\frac{\sigma - \frac{1}{2}}{\sigma + \frac{1}{2}} \right)^{\sigma - \frac{1}{2}}.$$

2. Let $\{\mu_k\}$ be the eigenvalues of the Dirichlet Laplacian $H_\Omega^D = -\Delta$ on an open domain $\Omega \subset \mathbb{R}^d$.

For any $\sigma \geq 1$, $\Lambda \geq 0$, $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ [Berezin '72]

$$\begin{aligned}\sum_k (\mu_k - \Lambda)_-^\sigma &\leq \frac{1}{(2\pi)^d} \int_\Omega dx \int_{\mathbb{R}^d} d\xi (|\xi|^2 - \Lambda)_-^\sigma \\ &\leq L_{\sigma,d}^{\mathrm{cl}} \mathrm{vol}(\Omega) \Lambda^{\sigma+\frac{d}{2}}.\end{aligned}$$

Proof. Let $\{\phi_k\}$ be an o.n. eigenbase H_Ω^D. Put $\phi_k \equiv 0$ on $\mathbb{R}^d \backslash \Omega$ and

$$\tilde{\phi}_k(\xi) = (2\pi)^{-d/2} \int_\Omega \phi_k(x) e^{i\xi x} dx.$$

Jensen's inequality ($\sigma \geq 1$, $\int_{\mathbb{R}^d} |\tilde{\phi}_k|^2 d\xi = 1$) gives

$$\begin{aligned} \sum_k (\mu_k - \Lambda)_-^\sigma &= \sum_k \left(\int_{\mathbb{R}^d} (|\xi|^2 - \Lambda) |\tilde{\phi}_k(\xi)|^2 d\xi \right)_-^\sigma \\ &\leq \int_{\mathbb{R}^d} (|\xi|^2 - \Lambda)_-^\sigma \sum_k |\tilde{\phi}_k(\xi)|^2 d\xi. \end{aligned}$$

Parsevals inequality w.r.t. $\{\phi_k\}$ in $L^2(\Omega)$ implies

$$\sum_k |\tilde{\phi}_k(\xi)|^2 = \int_\Omega |e^{-ix\xi}|^2 \frac{dx}{(2\pi)^d} = \int_\Omega \frac{dx}{(2\pi)^d}. \quad \square$$

3. The Legendre transformed $\widehat{f}(p)$ of a convex, non-negative function $f(t)$ on $\mathbb{R}_+$ is given by

$$\widehat{f}(p) = \sup_{t \geq 0} \left(pt - f(t)\right), \quad p \geq 0.$$

It reverses inequalities: $f(t) \leq g(t)$ for all $t \geq 0$ implies $\widehat{f}(p) \geq \widehat{g}(p)$ for all $p \geq 0$.

Note that

$$\begin{aligned} (\sum_k (\mu_k - x)_-)^\wedge(p) &= (p - [p])\mu_{[p]+1} + \sum_{k=1}^{[p]} \mu_k, \\ \left(cx^{1+\beta}\right)^\wedge(p) &= \frac{\beta p^{1+\beta^{-1}}}{(1+\beta)^{1+\beta^{-1}} c^{\beta^{-1}}}. \end{aligned}$$

We put $\sigma = 1$ in Berezin‘s inequality

$$\sum_k (\mu_k - \Lambda)_- \leq L^{\mathrm{cl}}_{1,d} \mathrm{vol}(\Omega) \Lambda^{1+\frac{d}{2}}$$

and apply the Legendre transformation for $x = \Lambda$ and $p = n \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=1}^{n} \mu_k &\geq n^{1+\frac{2}{d}} \left(L^{\mathrm{cl}}_{1,d} \mathrm{vol}(\Omega) \right)^{-\frac{2}{d}} \frac{d}{2} \left(1 + \frac{d}{2} \right)^{-1-\frac{2}{d}} \\
&\geq n^{1+\frac{2}{d}} \left(L^{\mathrm{cl}}_{0,d} \mathrm{vol}(\Omega) \right)^{-\frac{2}{d}} \frac{d}{2+d}
\end{aligned}$$

and recover a well-known result by Li and Yau.

4. The harmonic oscillator. Put $m = (m_1, \dots, m_d)$,

$$V(x) = \Lambda - \sum_{k=1}^{d} m_k^2 x_k^2, \quad \Lambda > 0, \quad m_k > 0.$$

Then the operator $H(V)$ has the eigenvalues

$$\lambda_\tau(V) = -\Lambda + \sum_{k=1}^{d} m_k(1 + 2\tau_k),$$

with $\tau = (\tau_1, \dots, \tau_d)$ and $\tau_k = 0, 1, 2, \dots$

For $\sigma, d = 1$ it holds $S^{\text{cl}}_{1,1}(V) = \frac{\Lambda^2}{4m_1}$ and

$$\begin{aligned} S_{1,1}(V) &= \sum_k (m_1(1+2k) - \Lambda)_- \\ &= m_1 \left(\frac{\Lambda^2}{(2m_1)^2} - t^2 \right) \end{aligned}$$

where $t = 1 + \left[\frac{\Lambda}{2m_1} - \frac{1}{2}\right] - \frac{\Lambda}{2m_1}$.

With the Lieb-Aizenman argument we get

$$S_{\sigma,1}(V) \leq S^{\mathsf{cl}}_{\sigma,1}(V), \quad \sigma \geq 1.$$

A straightforward generalisation to higher dimensions is much more involved and gives [De la Bretèche]

$$S_{\sigma,d}(V) \leq S^{\mathsf{cl}}_{\sigma,d}(V), \quad \sigma \geq 1.$$

The careful analysis of the same problem implies $R(\sigma, d) > 1$ for all $\sigma < 1$ [Helffer, Robert].

5. Alternatively, put $V(x) = W(x_1, \ldots, x_{d-1}) - m_d^2 x_d^2$. Integration in x_d and ξ_d gives

$$\begin{aligned} S^{\mathrm{cl}}_{\sigma,d}(V) &= \int\!\!\int \left(|\xi|^2 + m_d^2 x_d^2 - W(x')\right)_-^{\sigma} \frac{dx\,d\xi}{(2\pi)^d} \\ &= (2(\sigma+1)m_d)^{-1} S^{\mathrm{cl}}_{\sigma+1,d-1}(W). \end{aligned}$$

Moreover, $\lambda_{\tau',\tau_d}(V) = \lambda_{\tau'}(W(x')) + m_d(1 + 2\tau_d)$.

Evaluating the sum over $\tau_d \geq 0$ first, it follows that

$$\begin{aligned} S_{\sigma,d}(V) &= \sum_{\tau',\tau_d} (\lambda_{\tau'}(W) + m_d(1+2\tau_d))_-^{\sigma} \\ &= \sum_{\tau'} S_{\sigma,1}\left(\lambda_{\tau'}(W) - m_d^2 x_d^2\right) \\ &\leq \sum_{\tau'} S^{\mathrm{cl}}_{\sigma,1}\left(\lambda_{\tau'}(W) - m_d^2 x_d^2\right) \\ &\leq \sum_{\tau'} \int\!\!\int \left(|\xi_d|^2 + m_d^2 x_d^2 - \lambda_{\tau'}(W)\right)_-^{\sigma} \frac{dx_d\,d\xi_d}{(2\pi)} \\ &\leq (2(\sigma+1)m_d)^{-1} \sum_{\tau'} (-\lambda_{\tau'}(W))^{\sigma+1} \end{aligned}$$

and for any $\sigma \geq 1$ it holds

$$S_{\sigma,d}(V)/S^{\mathsf{cl}}_{\sigma,d}(V) \leq S_{\sigma+1,d-1}(W)/S^{\mathsf{cl}}_{\sigma+1,d-1}(W).$$

For $V = \Lambda - \sum_k m_k^2 x_k^2$ iteration gives

$$S_{1,d}(V) \leq S^{\mathsf{cl}}_{1,d}(V), \quad \sigma \geq 1.$$

In fact, this holds for all $V(x) = W(x') - m_d^2 x_d^2$!

6. We summarise

$$
\begin{aligned}
R(\sigma,1) &= 1 \quad \text{for} \quad \sigma \geq 3/2,\ d=1,\\
R(1/2,1) &= 2 \quad \text{for} \quad \sigma = 1/2,\ d=1,\\
R(\sigma,1) &\geq 2\left(\frac{\sigma-\frac{1}{2}}{\sigma+\frac{1}{2}}\right)^{\sigma-\frac{1}{2}} \quad \text{for} \quad \frac{1}{2}<\sigma<\frac{3}{2},\ d=1,\\
R(\sigma,d) &> 1 \quad \text{for} \quad \sigma<1,\ d\in\mathbb{N},\\
R(\sigma,2) &> 1 \quad \text{for} \quad \sigma<\sigma_0,\ \sigma_0\simeq 1.16,\ d=2.
\end{aligned}
$$

The Dirichlet Laplacian and the harmonic oscillator permitt a LTH estimate with the classical constant if $\sigma \geq 1$.

There exist certain explicite upper bounds on the constants $R(\sigma,d)$.

Lieb and Thirring posed the following

Conjecture:

In any dimension d there exists a finite critical value $\sigma_{\mathsf{cr}}(d)$, such that $R(\sigma, d) = 1$ for all $\sigma \geq \sigma_{\mathsf{cr}}(d)$.

In particular, one expects that $\sigma_{\mathsf{cr}}(d) = 1$ for $d \geq 3$, or

$$L_{1,d} = L^{\mathsf{cl}}_{1,d} \quad \text{for} \quad d \geq 3.$$

3. Lieb-Thirring Inequalities for Operator Valued Potentials

1. We consider a generalisation of LTh inequalities:

G is a separable Hilbert space, 1_G is the identity on G. $V : \mathbb{R}^d \to \mathcal{S}_\infty(G)$ is a compact s.-a. operator-valued fct.

We study the negative spectrum $\{\lambda_n(V)\}$ of the operator

$$H(V) = -\Delta \otimes 1_G - V(x) \quad \text{on} \quad L^2(\mathbb{R}^d) \otimes G.$$

In particular, we shall find bounds

$$S_{\sigma,d}(V) \le r(\sigma,d) S^{\mathrm{cl}}_{\sigma,d}(V)$$

of the eigenvalue moments

$$S_{\sigma,d}(V) = \mathrm{tr}_{L^2(\mathbb{R}^d)\otimes G} H^{\sigma}_{-}(V) = \sum_n (-\lambda_n(V))^{\sigma}$$

in terms of the classical counterparts

$$\begin{aligned} S^{\mathrm{cl}}_{\sigma,d}(V) &= \int\int \mathrm{tr}_G h^{\sigma}_{-}(\xi,x) \frac{dx d\xi}{(2\pi)^d} \\ &= L^{\mathrm{cl}}_{\sigma,d} \int \mathrm{tr}_G V_{-}^{\sigma+\frac{d}{2}}(x) dx \end{aligned}$$

where $h(\xi,x) = |\xi|^2 \otimes 1_G - V(x)$.

The constants $r(\sigma,d)$ should not depend on $\dim G$. Obviously

$$1 \le R(\sigma,d) \le r(\sigma,d).$$

2. Main results:

We confirm the first part of the conjecture by Lieb and Thirring with $\sigma_{\mathsf{cl}} \leq 3/2$.

Theorem 1. [A. Laptev, T. Weidl (Acta Math **184** (2000) 87-111)] *The identity*

$$R(\sigma, d) = r(\sigma, d) = 1$$

holds true for all $\sigma \geq 3/2$ *and all* $d \in \mathbb{N}$.

The most interesting case for applications to physics is $\sigma = 1$ and $d = 3$. Here the best know estimate so far was $R(1,3) \leq 5.24$. We show

Theorem 2. [D. Hundertmark, A. Laptev, T. Weidl (Invent math **140** 3 (2000) 693-704)] *The bounds*

$$R(\sigma, d) \leq r(\sigma, d) \leq \begin{cases} 4 & \textit{for} \quad \frac{1}{2} \leq \sigma < 1 \\ 2 & \textit{for} \quad 1 \leq \sigma < \frac{3}{2} \end{cases}$$

hold true in all dimensions $d \in \mathbb{N}$. *Moreover, if* $d = 1$ *then*

$$R\,(1/2, 1) = r\,(1/2, 1) = 2 \quad \textit{for} \quad \sigma = 1/2,$$
$$1 \leq R(\sigma, 1) \leq r(\sigma, 1) \leq 2 \quad \textit{for} \quad 1/2 < \sigma < 3/2.$$

3. Some Remarks:

The [LTh]-methods gives bounds for operator valued potentials with the same constants as in [LTh] for $\sigma > 1/2$ if $d = 1$ and for $\sigma > 0$ for $d \geq 2$.

It is not known whether $r(0, d)$ is finite for $d \geq 3$ (CLR inequality).

It is not known whether $r(\sigma, d) = R(\sigma, d)$ in general.

4. The proof of Theorem consists of two key elements:

Space dimension $d = 1$: We establish the identity

$$r(\sigma, 1) = 1 \quad \text{for all} \quad \sigma \geq 3/2.$$

This generalises $R(\sigma, 1) = 1$ for $\sigma \geq 3/2$ from scalar Schrödinger operators to Schrödinger operators with operator valued potentials .

We derive and apply a *trace formula* for matrix valued potentials for $\sigma = 3/2$ and use the AL-trick. Alternatively, Benguria and Loss found a proof based on a Darboux commutation method.

Space dimensions $d \geq 2$:
We iterate in the dimension.

$$
\begin{aligned}
S_{\sigma,d}(V) = \mathrm{tr}_{L^2(\mathbb{R}^d)\otimes G} H_-^\sigma = & \\
= \; & \mathrm{tr}_{L^2(\mathbb{R}^d)\otimes G} \left(-\frac{\partial^2}{\partial x_d^2} \otimes 1_G - \left(\Delta' \otimes 1_G + V(x'; x_d) \right) \right)_-^\sigma \\
\leq \; & \mathrm{tr}_{L^2(\mathbb{R})\otimes \tilde{G}} \left(-\frac{d^2}{dx_d^2} \otimes 1_{\tilde{G}} - W_-(x_d) \right)_-^\sigma = S_{\sigma,1}(W_-),
\end{aligned}
$$

where $x' = (x_1, \ldots, x_{d-1})$, Δ' is the Laplacian in the coordinates x' and $W(x_d)$ is the operator

$$
-\Delta' \otimes 1_G - V(x'; x_d) \quad \text{on} \quad \tilde{G} = L^2(\mathbb{R}^{d-1}) \otimes G.
$$

For $d = 1$, $\sigma \geq 3/2$ we apply the sharp LTH bound for the operator potential W_- and find

$$\begin{aligned} S_{\sigma,d}(V) &\leq \overbrace{L^{\mathsf{cl}}_{\sigma,1} \int \operatorname{tr}_{\tilde{G}} \quad \underbrace{W_-^{\sigma+\frac{1}{2}}(x_d)}_{(-\Delta'-V(\cdot;x_d))_-^{\sigma+\frac{1}{2}}} \quad dx_d}^{S^{\mathsf{cl}}_{\sigma,1}(W_-)} \\ &\leq L^{\mathsf{cl}}_{\sigma,1} \int S_{\sigma+\frac{1}{2},d-1}(V(\cdot;x_d))dx_d. \end{aligned}$$

We can continue this induction procedure and find

$$S_{\sigma,d}(V) \leq \underbrace{\prod_{k=0}^{d-1} L^{\mathsf{cl}}_{\sigma+\frac{k}{2},1}}_{L^{\mathsf{cl}}_{\sigma,d}} \int_{\mathbb{R}^d} V_+^{\sigma+\frac{d}{2}} dx = S^{\mathsf{cl}}_{\sigma,d}(V). \quad \square$$

4. Trace Formulae for matrix valued potentials

1. Put $G = \mathbb{C}^n$ and consider the system of ODE

$$-\left(d^2/dx^2 \otimes 1_G\right) y(x) - V(x)y(x) = k^2 y(x),\ x \in \mathbb{R}.$$

V is a smooth Hermitian valued matrix function with compact support.

For given $k \in \mathbb{C}\backslash\{0\}$ fix the $n \times n$ matrix-solutions

$$y(x) = F(x,k) = e^{ikx} 1_G \quad \text{as} \quad x \to +\infty,$$
$$y(x) = G(x,k) = e^{-ikx} 1_G \quad \text{as} \quad x \to -\infty.$$

The pairs of matrices $F(x,k)$, $F(x,-k)$ and $G(x,k)$, $G(x,-k)$ form full systems of independent solutions and

$$F(x,k) = G(x,k)B(k) + G(x,-k)A(k)$$

defines uniquely the matrix functions $A(k)$ and $B(k)$.

For $k \in \mathbb{R}$ we have

$$A(k)A^*(k) = 1_G + B(-k)B^*(-k)$$

and $|\det A(k)| \geq 1$.

2. The Buslaev-Faddeev-Zakharov trace formulae can be generalised to matrix potentials. For $V = V_+ \geq 0$ they read as follows

$$\begin{aligned} S^{\mathrm{cl}}_{\frac{1}{2},1}(V) &= -I_0 + S_{\frac{1}{2},1}(V) \\ S^{\mathrm{cl}}_{\frac{3}{2},1}(V) &= 3I_2 + S_{\frac{3}{2},1}(V) \\ S^{\mathrm{cl}}_{\frac{5}{2},1}(V) + \frac{1}{2}L^{\mathrm{cl}}_{\frac{5}{2},1}\int \mathrm{tr}_G\left(\frac{dV}{dx}\right)^2 dx &= -5I_4 + S_{\frac{5}{2},1}(V) \\ &\vdots \end{aligned}$$

Here is

$$I_j = \frac{1}{2\pi}\int_{\mathbb{R}} k^j \ln|\det A(k)|dk, \quad j = 0, 2, 4, \ldots$$

Because of $|\det A(k)| \geq 1$ we find $I_j \geq 0$.

3. Removing $3I_2$ from the second trace identity we claim

$$S_{3/2,1}(V) \leq S^{\mathrm{cl}}_{3/2,1}(V)$$

and $r(3/2,1) = 1$. This bound holds for indefinite V as well.

4. The first trace identity leads to the lower bound in

$$S^{\mathrm{cl}}_{\frac{1}{2},1}(V) \leq S_{\frac{1}{2},1}(V) \leq 2S^{\mathrm{cl}}_{\frac{1}{2},1}(V),$$

which holds only for $V \geq 0$.

5. If we apply the LTH bound $S_{1/2,1}(V) \leq 2S^{\text{cl}}_{1/2,1}(V)$ to the first trace identity, we get for $V = V_+ \geq 0$

$$\begin{aligned} I_0 &= S_{1/2,1}(V) - S^{\text{cl}}_{1/2,1}(V) \\ &\leq S^{\text{cl}}_{1/2,1}(V). \end{aligned}$$

Moreover, from $S_{5/2,1}(V) \leq S^{\text{cl}}_{5/2,1}(V)$ and the third trace identity it follows that

$$\begin{aligned} 5I_4 &= S_{\frac{5}{2},1}(V) - S^{\text{cl}}_{\frac{5}{2},1}(V) + \frac{1}{2}L^{\text{cl}}_{\frac{5}{2},1}\int \text{tr}_G\left(\frac{dV}{dx}\right)^2 dx \\ &\leq \frac{1}{2}L^{\text{cl}}_{\frac{5}{2},1}\int \text{tr}_G\left(\frac{dV}{dx}\right)^2 dx. \end{aligned}$$

Hölder's inequality $I_2^2 \leq I_0 I_4$ implies that

$$I_2^2 \leq \frac{1}{196} \left(\int \mathrm{tr}_G V \, dx \right) \cdot \left(\int \mathrm{tr}_G \left(\frac{dV}{dx} \right)^2 dx \right) =: R^2.$$

We insert this into the second trace formula and scale $\hbar \to 0$ back and find the upper bound in the inequality

$$0 \leq \underbrace{S^{\mathrm{cl}}_{\frac{3}{2},1}(V;\hbar)}_{O(\hbar^{-1})} - \underbrace{S_{\frac{3}{2},1}(V;\hbar)}_{O(\hbar^{-1})} \leq \underbrace{3R}_{O(1)}.$$

5. Polyharmonic operators

1. Consider the operators

$$H_l(V;\hbar) = \hbar^{2l}(-\Delta)^l - V(x), \quad l \in \mathbb{N}$$

on $L^2(\mathbb{R}^d)$. Let $\{\lambda_{n;l}(V;\hbar)\}_n$ be the negative eigenvalues of $H_l(V;\hbar)$.

We study the inequalities

$$S_{\sigma,d,l}(V;\hbar) \leq R(\sigma,d,l) S^{\mathrm{cl}}_{\sigma,d,l}(V;\hbar)$$

where

$$S_{\sigma,d,l}(V;\hbar) = \sum_n (-\lambda_{n;l}(V;\hbar))^\sigma$$

and

$$\begin{aligned} S^{\mathrm{cl}}_{\sigma,d,l}(V;\hbar) &= \int\int (h_l(\xi,x))^{\sigma}_{-} \frac{dx d\xi}{(2\pi\hbar)^d} \\ &= L^{\mathrm{cl}}_{\sigma,d,l}\hbar^{-d}\int V_+^{1+\kappa}dx \end{aligned}$$

with $h_l(\xi,x) = |\xi|^{2l} - V(x)$ and

$$L^{\mathrm{cl}}_{\sigma,d,l} = \frac{\Gamma(\sigma+1)\Gamma(\kappa+1)}{2^d\pi^{d/2}\Gamma(l\kappa+1)\Gamma(\kappa+\sigma+1)}, \quad \kappa = \frac{d}{2l}.$$

2. The LTH-inequality holds true if and only if

$$\begin{aligned} \sigma \geq 0 \quad &\text{for} \quad \kappa > 1, \\ \sigma > 0 \quad &\text{for} \quad \kappa = 1, \\ \sigma \geq 1-\kappa \quad &\text{for} \quad \kappa < 1. \end{aligned}$$

3. The LTH-inequality holds for *non-integer* values of l as well, except possibly for the critical case $\sigma_0 = 1 - \kappa > 0$, which has not been settled yet.

For $l \in \mathbb{N}$ and $\sigma_0 = 1 - \kappa > 0$ we find a two-sided estimate

$$\tilde{L}_{\sigma_0,d,l} \hbar^{-d} \int V \, dx \leq S_{\sigma_0,d,l}(V;\hbar) \leq L_{\sigma_0,d,l} \hbar^{-d} \int V_+ \, dx$$

with some positive, finite constants $\tilde{L}_{\sigma_0,d,l}$ and $L_{\sigma_0,d,l}$.

In analogy with $l = 1$ the weak and strong coupling behaviour suggest the conjecture

$$\tilde{L}_{\sigma_0,d,l} = L^{\mathrm{cl}}_{\sigma_0,d,l} \quad \text{and} \quad L_{\sigma_0,d,l} = \frac{\pi\kappa}{\sin \pi\kappa} L^{\mathrm{cl}}_{\sigma_0,d,l}.$$

4. For $\sigma > \max\{0, 1-\kappa\}$ the LTH bounds extend to operator valued potentials. This is not settled for $\sigma_0 = 1-\kappa$ if $\kappa < 1$ and for $\sigma = 0$ if $\kappa > 1$.

5. Constants in Lieb-Thirring inequalities for higher order operators are much less studied. No sharp values of the constants are known, not even in the dimension $d = 1$.