Standing generalized modulating pulse solutions for a nonlinear wave equation in periodic media

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Abstract

Standing modulating pulse solutions consist of a standing pulse-like envelope modulating an underlying spatially and temporarily oscillating carrier wave. Using spatial dynamics, invariant manifold theory and normal form theory for periodic systems we construct such solutions on large domains in time and space for a nonlinear wave equation with spatially periodic coefficients. Such solutions play an important role in theoretical scenarios where photonic crystals are used as optical storage.

Contents

1 Introduction

Photonic crystals consist of a dielectric material like glass with a periodic structure with a period comparable to the wavelength of light. They are suitable tools for the construction of all optical devices in photonics which is loosely speaking electronics with photons instead of electrons. There are various models describing the evolution of light and in particular in one-dimensional photonic crystals. The model mostly used in physics, cf. [4, Eq. 3.48, 3.49] is given by

$$\partial_x^2 u(x,t) = \partial_t^2((1+\alpha(x))u(x,t) + \beta(x)u(x,t)^3), \tag{1.1}$$

with x-dependent coefficients α and β satisfying $\alpha(x) = \alpha(x+L)$ and $\beta(x) = \beta(x+L)$ for an $L > 0, x \in \mathbb{R}, t \geq 0$, and $u(x,t) \in \mathbb{R}$. From a modelling point of view (see Appendix B) this quasi-linear equation is equivalent to the semi-linear model

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \rho(x)u(x,t) + \gamma r(x)u(x,t)^3, \tag{1.2}$$

where $\rho(x) = \rho(x+L)$, $r(x) = r(x+L)$, and where $\gamma = \pm 1$ has only been introduced for the formulation of our result. For notational purposes we will take $L = 2\pi$ w.l.o.g. in the following.

In principle, photonic crystals can be used as optical storage. Due to the periodicity, the linearized problem shows spectral gaps and curves of eigenvalues with horizontal tangencies, i.e., vanishing group velocities. In detail, $\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \rho(x)u(x,t)$ is solved by

$$e^{\mathrm{i}\omega_n(l)t} e^{\mathrm{i}lx} w_n(l,x), \tag{1.3}$$

with w_n satisfying the eigenvalue problem

$$(\partial_x + \mathrm{i}l)^2 w_n(l,x) - \rho(x) w_n(l,x) = -\omega_n^2(l) w_n(l,x), \qquad \text{with } w_n(l,x) = w_n(l,x+2\pi) \tag{1.4}$$

for all $l \in (-1/2, 1/2]$ and $n \in \mathbb{Z}/\{0\}$. The curves of eigenvalues $l \mapsto \omega_n(l)$ are ordered such that $\omega_{-n}(l) = -\omega_n(l)$ and $\omega_n(l) \le \omega_{n+1}(l)$. The spectral picture shows horizontal tangencies for certain n's at Bloch wave numbers $l \in \{0, \pm 1/2\}$, i.e., vanishing group velocity occurs, see Figure 1.

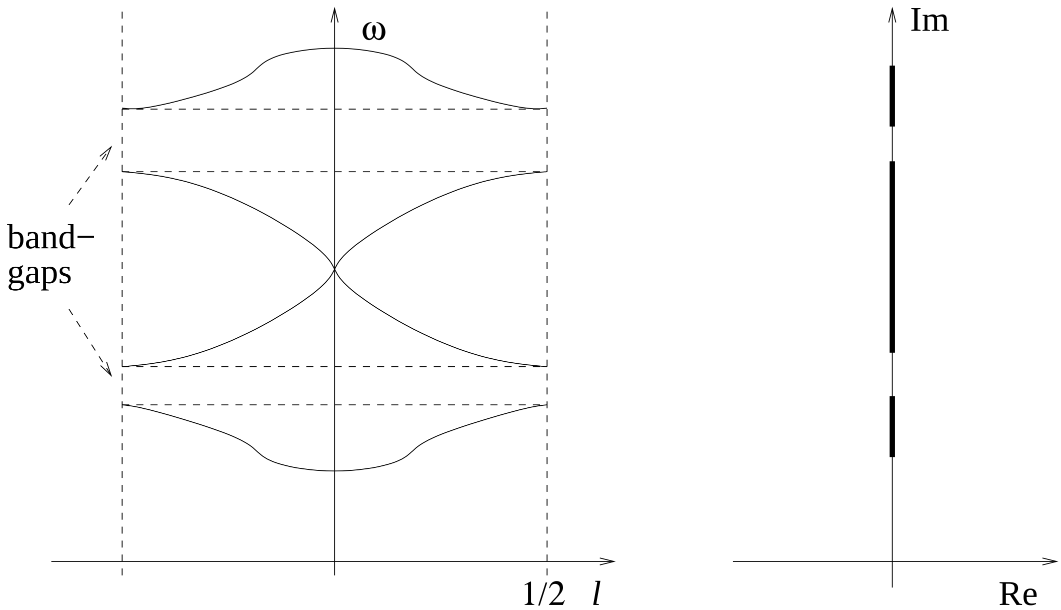

Figure 1: The eigenvalue problem of $(\partial_x + \mathrm{i}l)^2 w_n(l,x) - \rho(x) w_n(l,x) = -\omega_n^2(l) w_n(l,x)$ with periodic boundary conditions $w_n(l, x + 2\pi) = w_n(l,x)$ possesses countably many eigenvalues $-\omega_n^2(l)$ for fixed l. Here the first four curves $l \mapsto \omega_n(l)$ for a possible situation are shown. The right panel shows the spectrum with the spectral gaps as a subset of $\mathbb{C}$. In the left panel horizontal tangencies at $l = 0$ and $l = \pm 1/2$ can be seen.

Hence, it is expected that a localized wave packet of light with a suitable chosen carrier wave will not propagate in the interior of the photonic crystal, see Figure 2.

Therefore, it is the purpose of this paper to construct such standing modulating pulse solutions rigorously for the second of the above models, namely the semilinear wave equation (1.2). It cannot be expected that in general such solutions do exist with finite energy according to the non-persistence of breathers result for nonlinear wave equations in homogeneous media [5, 1], i.e. in case $\rho = r = 1$. However, generalized breather solutions, i.e. localized standing waves with small tails for $|x| \to \infty$, do exist. In the homogeneous situation such solutions have been constructed in [10] with the help of spatial dynamics, invariant manifold theory and normal form theory. In general for reasons explained below

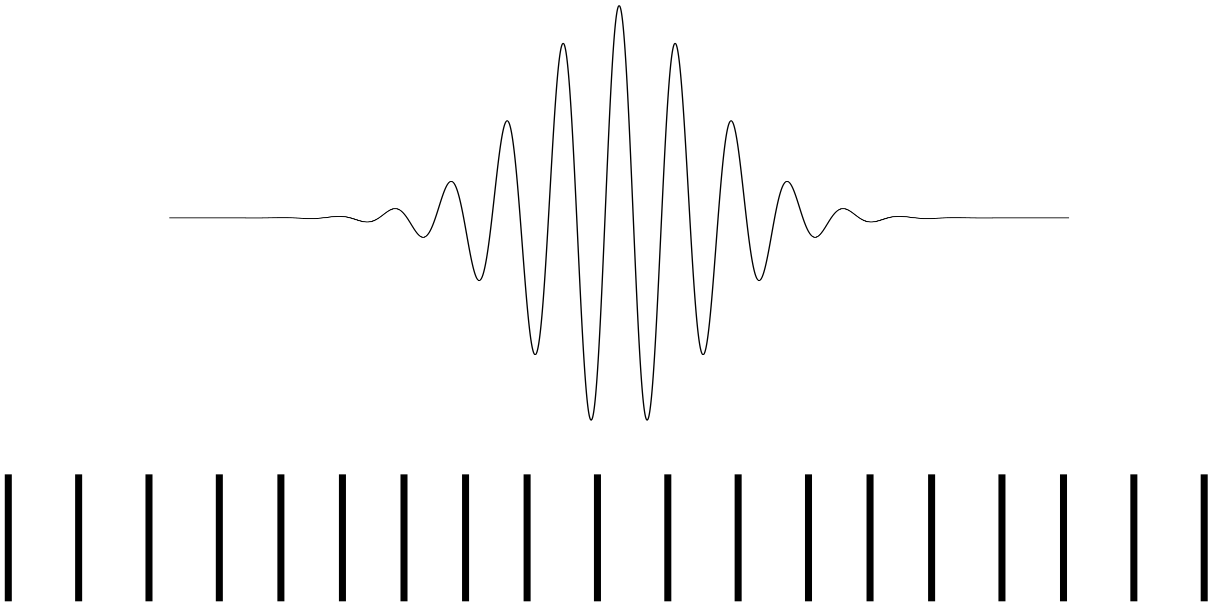

Figure 2: A standing light pulse (line) in a photonic crystal (solid bars). The wavelength of the carrier wave and of the photonic crystal are of a comparable order.

such solutions can only be constructed on large, but finite, intervals of $\mathbb{R}$, cf. [11, 12]. Our paper is based on these three papers and is as already said devoted to the construction of these generalized breather solutions in periodic media.

Since our methods stem from various fields it was not possible to use the standard notation in all cases.

1.1 The method

The analysis which is used in order to prove the existence of generalized breathers is changed substantially from the spatially homogeneous to the spatially periodic case. For periodic media, as considered here, the spatial dynamics formulation is no longer autonomous, but periodic. A more serious consequence is the resulting spectral situation. In addition to the infinitely many eigenvalues, resp. Floquet exponents, on the imaginary axis there can be infinitely many Floquet exponents off the imaginary axis, see Figure 4. Moreover, normal form transforms and the non-resonance conditions have to be adjusted to the periodic situation.

We start by rewriting (1.2) in spatial dynamics form

$$\partial_x^2 u(x,t) = \partial_t^2 u(x,t) + \rho(x)u(x,t) - \gamma r(x)u(x,t)^3, \tag{1.5}$$

i.e., $x \in \mathbb{R}$ is considered as evolutionary variable and the phase space consists of $2\pi/\omega$-time periodic solutions. Equation (1.5) is non-autonomous due to the 2π-periodic coefficients $\rho = \rho(x)$ and $r = r(x)$.

Breather solutions satisfy $\lim_{x\to\pm\infty} u(x,t) = 0$, i.e., such solutions are homoclinic to the origin w.r.t. the evolutionary variable x. They lie in the intersection of the stable and unstable manifold of the origin. Very often they are constructed as bifurcating solutions from the origin via center manifold theory. In order to find the dimension of these manifolds we consider the linearized problem

$$\partial_x^2 u(x,t) = \partial_t^2 u(x,t) + \rho(x) u(x,t), \tag{1.6}$$

first. Due to the periodic boundary conditions w.r.t. t it is solved by

$$u(x,t) = \mathrm{e}^{\mathrm{i} m\omega t} \mathrm{e}^{\lambda_{m,\pm} x} p_{m,\pm}(x), \tag{1.7}$$

with $p_{m,\pm}(x) = p_{m,\pm}(x + 2\pi)$ for $m \in \mathbb{Z}$. The $\lambda_{m,\pm}$s are called Floquet exponents and they are only unique up to integer multiples of $2\pi\mathrm{i}$.

In order to compute the spectrum we use the fact that there is a one-to-one correspondence between the spectral pictures drawn in Figure 1 and Figure 4. This is obvious by comparing the representations of solutions of the linearized temporal problem in (1.3) and of solutions of the linearized spatial problem in (1.7). When an integer multiple of the basic temporal wave-number ω falls into a spectral gap drawn in Figure 1 there are two Floquet exponents off the imaginary axis in Figure 4. In the other case the Floquet exponents are on the imaginary axis. See also Section 2.

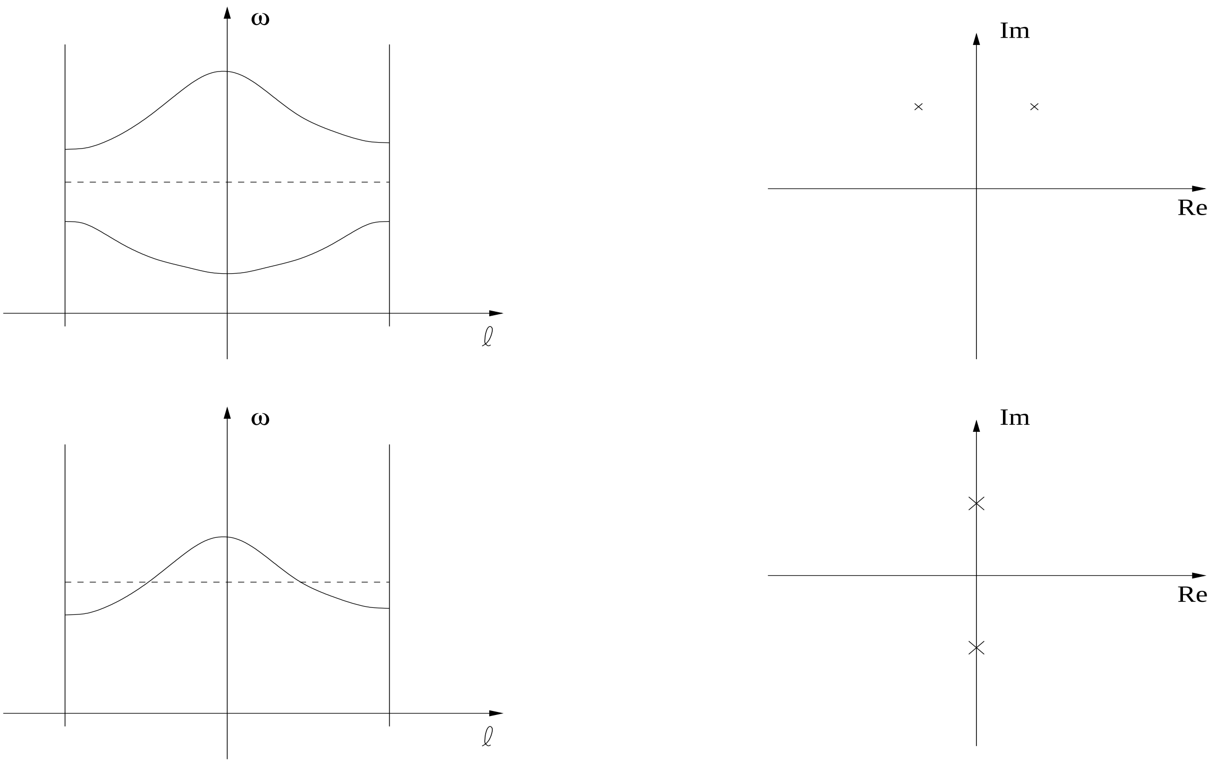

Figure 3: Curves of eigenvalues of the temporal formulation (left panels) and corresponding Floquet exponents of the spatial formulation (right panels). If the dotted line $l \mapsto m\omega$ "falls into a spectral gap" then two Floquet exponents lie off the imaginary axis (top panels). Otherwise both are on the imaginary axis (bottom panels).

For continuous or even smoother ρ the spectral gaps in

$$\{\omega_n(l) \mid l \in (-1/2, 1/2],\ n \in \mathbb{Z} \setminus \{0\}\} \subset \mathbb{R}$$

become smaller as n increases and, therefore, integer multiples $m\omega$ of ω in general do not fall into spectral gaps. In detail, according to [6, 17], the size of the gaps which are found at $\omega \sim n$ decays at least with $1/n^{s+1}$ for $n \to \infty$ if $\rho \in C_b^s$, i.e., the more regular ρ is the faster the gaps close. Hence, in contrast to classical applications of spatial dynamics and center manifold theory, which has been started with [16], in every neighborhood of the imaginary axis there are infinitely many Floquet exponents and so the problem cannot be reduced via center manifold theory to a finite dimensional one. In the homogeneous case there are no spectral gaps at all and so close to the bifurcation point, except for two, all eigenvalues lie on the imaginary axis. In the spatially periodic case there are more possibilities. A typical example of a spectral distribution is drawn in Figure 4.

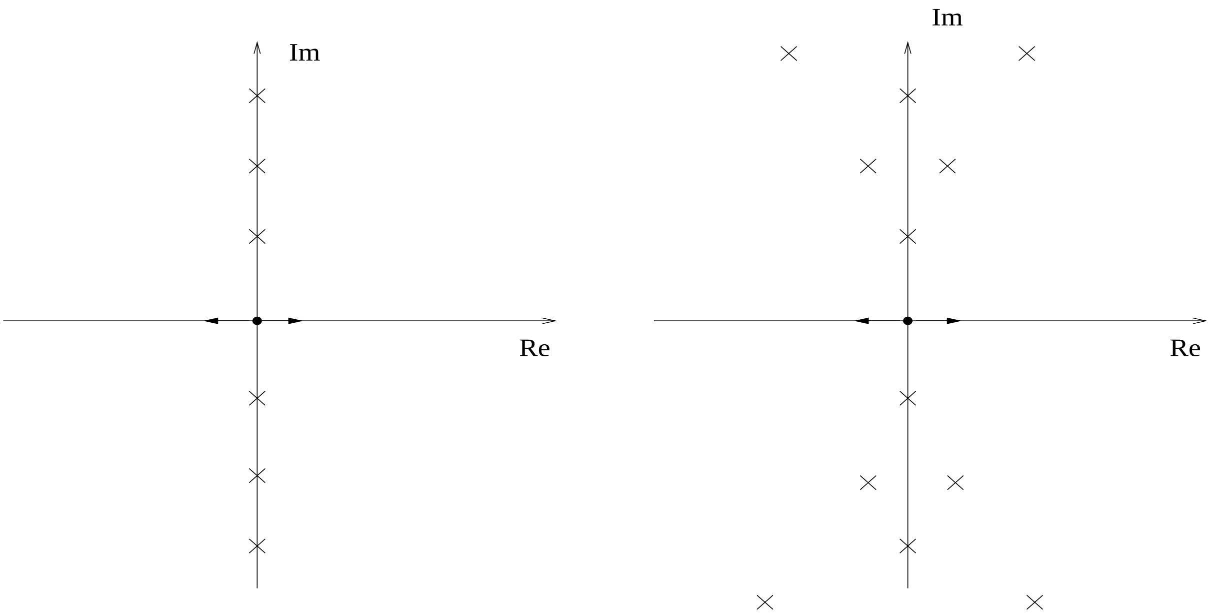

Figure 4: Examples of an eigenvalue and of a Floquet exponent distribution corresponding to the spatial dynamics formulation in the homogeneous case (left) and the spatially periodic case (right).

Although the spectral pictures in the spatial homogeneous and the spatial periodic case look very different the overall idea of [10] for the construction of generalized breather solutions still can be used.

As already explained above, in order to find standing spatially localized modulating pulse solutions of Equation (1.2) we have to construct homoclinic solutions of the spatial dynamics formulation (1.5). We fix a temporal wavenumber ω_0 such that ω_0 is located at the edge of a gap in Figure 3. In this case the associated Floquet exponents are zero or $\pm\mathrm{i}/2$ where for expository reasons we restrict to the first case in the following. By shifting ω into this spectral gap, the two zero Floquet exponents leave the imaginary axis along the real axis as drawn in Figure 4. The distance of $\varepsilon^2 = |\omega^2 - \omega_0^2|$ defines a small bifurcation parameter $0 < \varepsilon \ll 1$.

Although for small $\varepsilon > 0$ there are still infinitely many eigenvalues arbitrary close to the imaginary axis, the problem can be reduced approximately to a two-dimensional one by the following procedure. See Section 4 for more details. We split $u = q \oplus w$ where q

corresponds to the two zero Floquet exponents and w to the rest, i.e., we write (1.5) as

$$\partial_x q = B_1 q + F(q,w), \tag{1.8}$$

$$\partial_x w = Lw + G(q,w) + H(q), \tag{1.9}$$

with B_1 and L being linear operators and F, G, H being nonlinear operators. We split the nonlinear terms in the second equation such that $G(q,0) = 0$. If $H(q)$ vanishes, $\{w = 0\}$ is an invariant subspace and

$$\partial_x q = B_1 q + F(q,0) \tag{1.10}$$

is a reduced two-dimensional system which possesses two solutions which are homoclinic to the origin. See Section 5 for a detailed discussion of (1.10). The bifurcating solutions q which are constructed via (1.10) will be of order $\mathcal{O}(\varepsilon)$.

Obviously, $H(q) = \mathcal{O}(|q|^3) \neq 0$ originally, however, by a number of normal form transforms it can be made small. A term $\mathcal{O}(|q|^{N-1})$ can be eliminated in the equations for w by a normal form transform if the non-resonance condition

$$\lambda_j \notin \mathrm{i}\mathbb{Z} \tag{1.11}$$

is satisfied for $\omega = \omega_0$ and $j \in \{3, 5, \ldots, N-1\}$, see Section 4 for details. This assumption holds if for $j \in \{3, 5, \ldots, N-1\}$ the integer multiples $j\omega_0$ of the basic wave number ω_0 doesn't hit a band edge at $l = 0$.

As already explained by the non-persistence result for breathers, even if there is no resonance, it cannot be expected that the normal form transforms converge for $N \to \infty$, since this would lead to true breather solutions. Hence only the persistence of the bifurcating homoclinic solutions which have been found for $H = 0$ as generalized breather solution can be expected. These generalized breather solutions will be in the intersection of center-stable and the center-unstable manifold of the origin. Since the center-stable manifold also includes central modes via w polynomial growth of these modes is possible and so estimates for all $x \in \mathbb{R}$ cannot be expected. However, in case $H(q) = \mathcal{O}(|q|^N)$ for an $N \in \mathbb{N}$ fixed it can be shown that $w = \mathcal{O}(\varepsilon^{N-1})$ for all $x \in [-\varepsilon^{3-2N}, \varepsilon^{3-2N}]$. For more details see Sections 6 and 7. Therefore, the existence of generalized modulating pulse solutions, i.e., pulse solutions with small tails on large, but finite, intervals can finally be shown. See Figure 5.

In order to prove the persistence of the bifurcating homoclinic solutions which have been found for $H = 0$, also in case $H(q) = \mathcal{O}(|q|^N)$ for an $N \in \mathbb{N}$ fixed we use the reversibility of the spatial dynamics formulation, i.e. the invariance of (1.5) under $(x, u) \mapsto (-x, u)$. In order to have this invariance the coefficient functions ρ and r have to be even w.r.t. x, i.e., $\rho(x) = \rho(-x)$ and $r(x) = r(-x)$. See Sections 3.2 and 8.

1.2 The result

Motivated by the explanations of the previous section our result is as follows.

Theorem 1.1. *Let ρ and r be 2π-periodic, even functions in H^1_{per}. Assume that there are natural numbers n_0 and N such that for the linearization of Equation* (1.2) *there is* *a band-gap which begins or ends at* $\omega_{n_0}(0)$ *and that for* $|j| < N$ *the odd integer multiples* $j\omega_{n_0}(0)$ *of the basic wave number* $\omega_0 = \omega_{n_0}(0)$ *hit no other band edge at* $l = 0$.

Then under the validity of some non-degeneracy condition (cf. Remark 1.2) there exist an $\varepsilon_0 > 0$ *and a* $C > 0$ *such that either for* $\gamma = 1$ *or* $\gamma = -1$ *and either for* $\omega^2 - \omega_0^2 = \varepsilon^2$ *or* $\omega^2 - \omega_0^2 = -\varepsilon^2$ *the following holds. For all* $\varepsilon \in (0, \varepsilon_0)$ *Equation* (1.2) *possesses generalized modulating standing pulse solutions with period* $2\pi/\omega$*, i.e., there are solutions* $u : [-\varepsilon^{3-2N}, \varepsilon^{3-2N}] \times \mathbb{R} \to \mathbb{R}$ *of* (1.2) *which satisfy*

$$u(x,t) = u(-x,t), \quad u(x,t) = u\left(x, t + \frac{2\pi}{\omega}\right), \quad \forall x \in [-\varepsilon^{3-2N}, \varepsilon^{3-2N}], \quad \forall t \in \mathbb{R},$$

$$\sup_{x \in [-\varepsilon^{3-2N}, \varepsilon^{3-2N}]} |u(x,t) - h(x,t)| \leq C\varepsilon^{N-1},$$

where

$$\lim_{|x| \to \infty} h(x,t) = 0, \quad \forall t \in \mathbb{R},$$

and

$$\sup_{x,t \in \mathbb{R}} |h(x,t) - h_{\mathrm{app}}(x,t)| \leq C\varepsilon^2,$$

where

$$h_{\mathrm{app}}(x,t) = \varepsilon\gamma_1 \operatorname{sech}(\varepsilon\gamma_2 x) w_{n_0}(0,x) \mathrm{e}^{\mathrm{i}\omega_0 t} + c.c., \tag{1.12}$$

with constants γ_1, γ_2 *defined in Section 5 and a* 2π*-periodic function* $w_{n_0}(0,\cdot)$ *defined in* (1.4)*. The generalized modulating standing pulse solutions are* C^0 *w.r.t.* x *and* C^s *w.r.t.* t *for arbitrary* $s \in \mathbb{N}$.

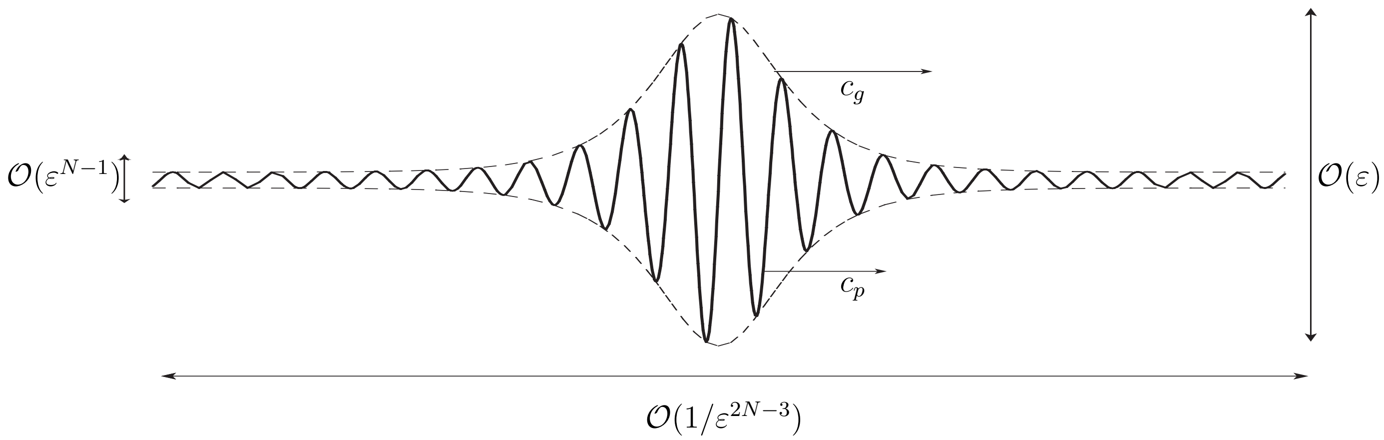

Figure 5: A generalized modulating pulse solution as constructed in Theorem 1.1 with $\mathcal{O}(\varepsilon^{N-1})$ tails existing for x in an interval of length $\mathcal{O}(1/\varepsilon^{2N-3})$ with an envelope modulating an underlying carrier wave advancing with phase velocity c_p. For the group velocity we have $c_g = 0$, but see also Remark 1.2.

Before we start to prove this result we close the introduction with a number of remarks.

Remark 1.2. *For all* $n_0 \in \mathbb{N}$ *and* $l_0 \in (-1/2, 1/2]$ *solutions of* (1.2) *can be approximated via the ansatz*

$$u(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) w_{n_0}(l_0, x) \mathrm{e}^{\mathrm{i} l_0 x} \mathrm{e}^{\mathrm{i}\omega_{n_0}(l_0) t} + \mathrm{c.c.},$$

with amplitude $A(X,T) \in \mathbb{C}$*, group velocity* $c_g = \omega'_{n_0}(l_0) \in \mathbb{R}$*, and* $0 < \varepsilon \ll 1$ *a small perturbation parameter by a NLS-equation*

$$\partial_T A = -\mathrm{i}\frac{\omega''_{n_0}(l_0)}{2}\partial_X^2 A + \mathrm{i}\gamma_{n_0}(l_0)\gamma A|A|^2 \tag{1.13}$$

where $c_g = 0$ *for* $l_0 = 0, \pm 1/2$*. The coefficient* $\gamma_{n_0}(l_0)$ *is given by*

$$\gamma_{n_0}(l_0) = \frac{3}{2\pi\omega_{n_0}(l_0)} \int_0^{2\pi} r(x)|w_{n_0}(l_0,x)|^4 dx.$$

The function w_{n_0} *is defined in* (1.4)*. The NLS-equation possesses pulse solutions* $A(X,T) = \tilde{A}(X)\mathrm{e}^{\mathrm{i}\tilde{\omega}T}$ *of the form displayed in* (1.12) *if* $\omega''_{n_0}(l_0)\gamma_n(l_0)\gamma < 0$*. In [3] an approximation result has been established that guarantees that solutions of* (1.2) *can be approximated on an* $\mathcal{O}(\varepsilon^{-2})$ *time scale via general solutions of this NLS-equation. The non-degeneracy condition for the validity of Theorem 1.1 is:* $\gamma_{n_0}(l_0) \neq 0$*.*

Remark 1.3. *If we take the solutions of Theorem 1.1 for* $t = 0$ *as initial condition for* (1.2)*, and take arbitrary initial conditions outside of the interval* $[-\varepsilon^{3-2N}, \varepsilon^{3-2N}]$ *then due to the fact that we have a finite maximal speed of propagation* $c_{\max} = \mathcal{O}(1) > 0$ *for* (1.2)*, the solutions of Theorem 1.1 exist for* (1.2) *also for all*

$$\{(x,t) \mid t \in [0, \varepsilon^{3-2N}/c_{\max}], \quad x \in [-\varepsilon^{3-2N} + c_{\max}t, \varepsilon^{3-2N} - c_{\max}t]\}.$$

Hence the modulated pulse solutions can be seen much longer than on the $\mathcal{O}(1/\varepsilon^2)$*-time scale guaranteed by the approximation theorem given in [3].*

Remark 1.4. *If the non resonance condition* (1.11) *is satisfied for all* $j \in \mathbb{Z}$*, then* N *can be chosen arbitrarily large, but has to be fixed. However, we have* $\varepsilon_0 \to 0$ *if* $N \to \infty$ *is chosen. The result of [11] has been improved in [12] to exponentially small tails and exponentially long time intervals w.r.t.* ε*. It is not obvious that this last result can be transferred to the spatially periodic case. It is also not clear that the method from Reference [10] which uses the Hamiltonian structure of* (1.2) *to obtain* $x \in \mathbb{R}$ *can be transferred to the spatially periodic case.*

Remark 1.5. *If all integer multiples of the basic temporal wave-number* ω *fall in a suitable way into a spectral gap in Figure 1 the center manifold reduction can be applied and modulating pulse solutions for all* $x \in \mathbb{R}$ *of* (1.2) *with* $\lim_{|x|\to\infty} u(x,t) = 0$ *can be shown. However, in this case* ρ *has to be less regular. See [2].*

Remark 1.6. *The regularity assumption* $\rho \in H^1_{\mathrm{per}}$ *is used in the proof of the subsequent Lemma 3.5. This lemma can be proved directly in case of step functions. Hence Theorem 1.1 is also true in case of periodic step functions.*

Remark 1.7. *In case when ω_0 hits a band edge at $l_0 = 1/2$ Theorem 1.1 has to be modified as follows:*

Let ρ and r be 2π-periodic, even functions in H^1_{per}. Assume that there are natural numbers n_0 and N such that for the linearization of Equation (1.2) *there is a band-gap which begins or ends at $\omega_{n_0}(1/2)$ and that for $|j| < N$ the odd integer multiples $j\omega_{n_0}(1/2)$ of the basic wave number $\omega_0 = \omega_{n_0}(1/2)$ hit no other band edge at $l = 1/2$.*

Then we have exactly the same conclusions as in Theorem 1.1 except that (1.12) *has to be replaced by*

$$h_{\text{app}}(x,t) = \big(\varepsilon\gamma_1 \operatorname{sech}(\varepsilon\gamma_2 x) w_{n_0}(1/2, x) e^{ix/2} e^{i\omega_0 t} + c.c.\big) \tag{1.14}$$

See Figure 6.

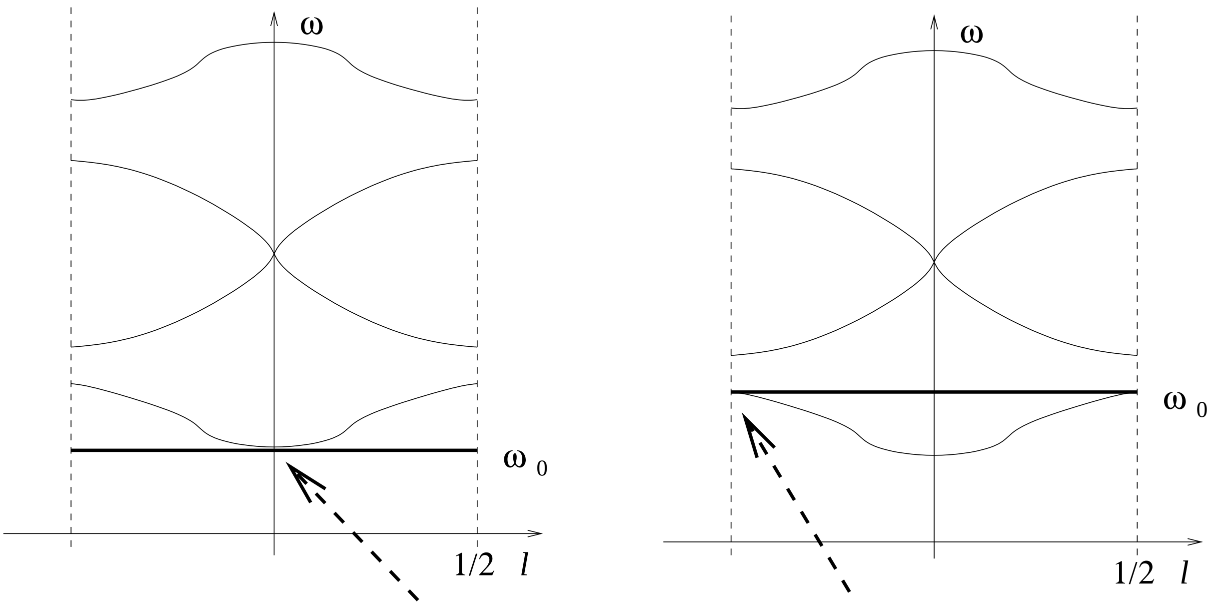

Figure 6: The left panel shows the situation in Theorem 1.1. The right panel shows the situation in Remark 1.7

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2 The spatial dynamics formulation

We write (1.2) as an evolutionary system w.r.t. $x \in \mathbb{R}$ and obtain

$$\partial_x^2 u(x,t) = \partial_t^2 u(x,t) + \rho(x) u(x,t) - \gamma r(x) u^3(x,t). \tag{2.1}$$

in the space of time-periodic solutions $u(x,t) = u(x,t+2\pi/\omega)$. Hence, we use Fourier series $u(x,t) = \sum_{m\in\mathbb{Z}} u_m(x) e^{\mathrm{i}m\omega t}$ leading to the system of countably many ODEs

$$\partial_x^2 u_m(x) = -m^2\omega^2 u_m(x) + \rho(x) u_m(x) - \gamma r(x) g_m(x), \qquad m \in \mathbb{Z}, \tag{2.2}$$

where

$$g_m(x) = \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}\\ n_1+n_2+n_3=m}} u_{n_1}(x) u_{n_2}(x) u_{n_3}(x), \qquad m \in \mathbb{Z}. \tag{2.3}$$

2.1 Symmetries

The dimension of the problem can be reduced by using the symmetries of the problem. Real-valued solutions of (2.1) satisfy $u_m = \overline{u_{-m}}$. System (2.1) is invariant under the transform $S : (t,u) \mapsto (-t,-u)$. Solutions invariant under this transform satisfy $u_m = -u_{-m}$. According to the fact that we have a cubic nonlinearity also the space of solutions whose even coefficients vanish, i.e. $u_{2m} = 0$, is an invariant subspace. Therefore, the intersection of all these subspaces

$$\hat{X} = \{(u_m)_{m\in\mathbb{Z}} \ : \ \operatorname{Re} u_m = 0, u_{2m} = 0, u_m = -u_{-m}, m \in \mathbb{Z}\}$$

is also invariant. In the following we restrict our analysis to those solutions of (2.2) which are in $\hat{X}$ for fixed x, i.e., it is sufficient to consider $m \in \mathbb{N}_{odd} = \{1,3,5,\ldots\}$ where $u_m \in \mathrm{i}\mathbb{R}$. Hence, u_m could be replaced by $u_m = \mathrm{i}\tilde{u}_m$ where $\tilde{u}_m$ satisfies (2.2) with the opposite sign in front of the nonlinear terms, i.e., the subsequent equations for the u_m have the properties of real-valued equations.

2.2 Linear theory

The next step in the application of invariant manifold theory is the analysis of the linear system. The linear part of the spatial dynamics system (2.2) decouples into infinitely many linear second order ODEs with periodic coefficients, namely

$$\partial_x^2 u_m(x) = -m^2\omega^2 u_m(x) + \rho(x) u_m(x). \tag{2.4}$$

Equation (2.4) is written as first order system, namely

$$\begin{aligned} \partial_x u_m(x) &= v_m(x), \\ \partial_x v_m(x) &= -m^2\omega^2 u_m(x) + \rho(x) u_m(x). \end{aligned} \tag{2.5}$$

The fundamental matrix of (2.5) is denoted with $\Phi_m = \Phi_m(x)$ where $\Phi_m(0) = I$. Floquet's theorem [6] shows that

$$\Phi_m(x) = \tilde{P}_m(x) e^{x\tilde{M}_m}$$

with periodic matrix $\tilde{P}_m(x) = \tilde{P}_m(x+2\pi)$ and an x-independent matrix $\tilde{M}_m$. Note that $\tilde{M}_m$ is not unique according to $\mathrm{e}^{2\pi\mathrm{i}n} = 1$ for $n \in \mathbb{Z}$. The eigenvalues of $\tilde{M}_m$ are called Floquet exponents. The eigenvalues of the so called monodromy matrix $\tilde{C}_m = e^{2\pi\tilde{M}_m}$ are called Floquet multipliers.

For our special system, cf. [6], we have two Floquet multipliers σ_1 and σ_2 satisfying $\sigma_1\sigma_2 = 1$ and which are either real or complex conjugated. As a consequence we find

$$\sigma \in \tilde{\Sigma}_1^m \cup \tilde{\Sigma}_2^m \cup \tilde{\Sigma}_3^m \cup \tilde{\Sigma}_4^m$$

where

$$\begin{aligned}
\tilde{\Sigma}_1^m &= \{\sigma \in \mathbb{R} \mid \sigma \in (-\infty,-1) \cup (-1,0)\},\\
\tilde{\Sigma}_2^m &= \{\sigma \in \mathbb{R} \mid \sigma \in \{-1,1\}\},\\
\tilde{\Sigma}_3^m &= \{\sigma \in \mathbb{C} \mid \sigma = e^{i\varphi},\ \varphi \in (-\pi,0) \cup (0,\pi)\},\\
\tilde{\Sigma}_4^m &= \{\sigma \in \mathbb{R} \mid \sigma \in (0,1) \cup (1,\infty)\}.
\end{aligned}$$

Thus the associated Floquet exponents λ_1 and λ_2 satisfy

$$\lambda \in \tilde{\Sigma}_1^e \cup \tilde{\Sigma}_2^e \cup \tilde{\Sigma}_3^e \cup \tilde{\Sigma}_4^e$$

where

$$\begin{aligned}
\tilde{\Sigma}_1^e &= \{\lambda \in \mathbb{C} \mid \lambda = i/2 + x,\ x \in (-\infty,0) \cup (0,\infty)\},\\
\tilde{\Sigma}_2^e &= \{\lambda \in \mathbb{C} \mid \lambda \in \{i/2,0\}\},\\
\tilde{\Sigma}_3^e &= \{\lambda \in \mathbb{C} \mid \lambda = iy,\ y \in (-1/2,0) \cup (0,1/2)\},\\
\tilde{\Sigma}_4^e &= \{\lambda \in \mathbb{C} \mid \lambda = x,\ x \in (-\infty,0) \cup (0,\infty)\}.
\end{aligned}$$

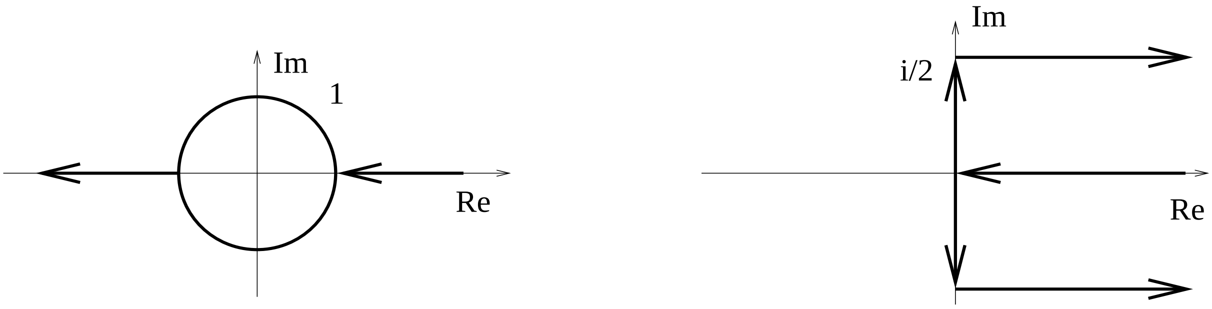

Figure 7: The left panel shows two curves of Floquet multipliers. The right panel shows the associated curves of Floquet exponents. These curves correspond in Figure 6 to a monotonic increase of ω_0 where the point when the Floquet exponents are 0 is drawn in the left panel, whereas the point when the Floquet exponents are $-i/2$ and $i/2$ which are identified is drawn in the right panel. Due to the fact that we have horizontal tangencies in Figure 6 the Floquet exponents 0 and $i/2$ have algebraic multiplicity two, but geometric multiplicity one, i.e., a Jordan-block in the spatial dynamics formulation will occur.

2.3 Doubling the period

Considering the problem artificially in a 4π-periodic setting instead of a 2π-periodic setting reduces the possible cases of Floquet exponents by one. Since $(e^{i\pi})^2 = 1$ there are no longer Floquet multipliers σ with negative real part, i.e. we have

$$\sigma \in \Sigma_2^m \cup \Sigma_3^m \cup \Sigma_4^m$$

where

$$\begin{array}{rcl} \Sigma_2^m & = & \{\sigma \in \mathbb{C} \mid \sigma \in \{1\}\}, \\ \Sigma_3^m & = & \{\sigma \in \mathbb{C} \mid \sigma = e^{\mathrm{i}\varphi},\ \varphi \in (-\pi/2, 0) \cup (0, \pi/2]\}, \\ \Sigma_4^m & = & \{\sigma \in \mathbb{R} \mid \sigma \in (0,1) \cup (1, \infty)\}. \end{array}$$

Thus the associated Floquet exponents λ_1 and λ_2 satisfy

$$\lambda \in \Sigma_2^e \cup \Sigma_3^e \cup \Sigma_4^e$$

where

$$\begin{array}{rcl} \Sigma_2^e & = & \{\lambda \in \mathbb{C} \mid \lambda \in \{0\}\}, \\ \Sigma_3^e & = & \{\lambda \in \mathbb{C} \mid \lambda = \mathrm{i}y,\ y \in (-1/4, 0) \cup (0, 1/4]\}, \\ \Sigma_4^e & = & \{\lambda \in \mathbb{C} \mid \lambda = x,\ x \in (-\infty, 0) \cup (0, \infty)\}. \end{array}$$

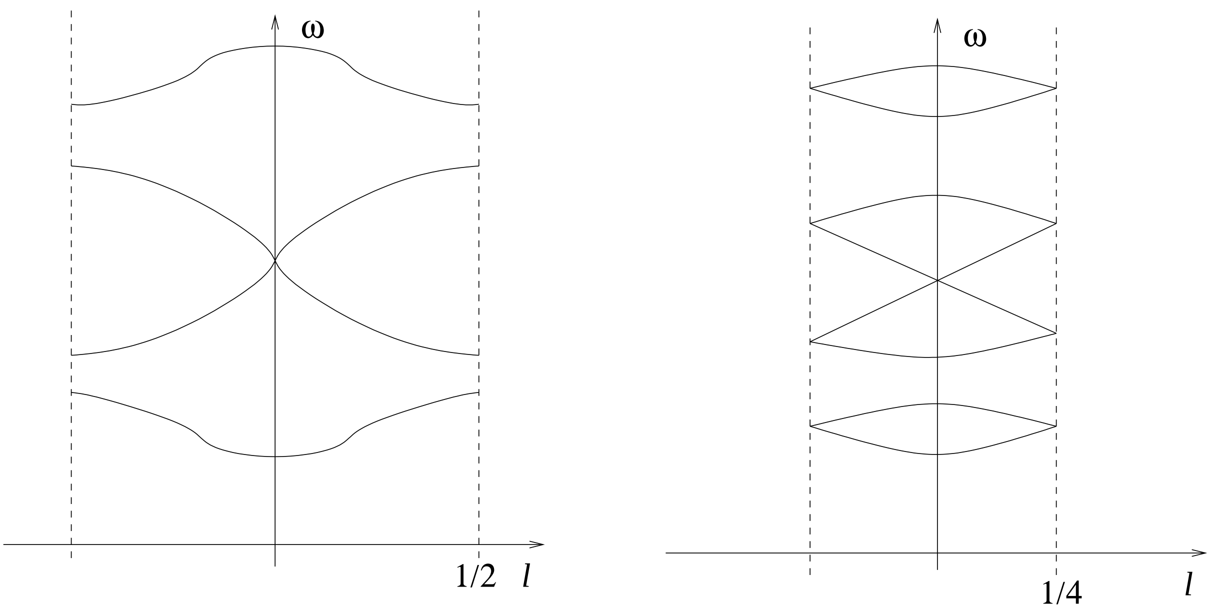

Figure 8: The left panel shows a sketch of the curves of eigenvalues in the 2π-periodic setting. The right panel shows a sketch of the associated curves of eigenvalues in the artificial 4π-periodic setting. Horizontal tangencies can only occur at $l = 0$. They correspond in the spatial dynamics formulation to Jordan blocks.

By choosing the 4π-periodic setting the reversibility becomes simpler, but the normal form transforms become a little bit more involved. Floquet's theorem then shows for the fundamental matrix $\Phi = \Phi(x)$ of (2.5) that

$$\Phi_m(x) = P_m(x) e^{xM_m}$$

but now with a periodic matrix $P_m(x) = P_m(x + 4\pi)$ and a x-independent matrix M_m.

3 A reversible change of variables

In this section we make a change of variables which makes the linear part autonomous and preserves the reversibility. In order to do so we rewrite (2.2) as first order system

$$\partial_x U_m = \Lambda_m U_m + N_m(U), \quad m \in \mathbb{N}_{odd}, \tag{3.1}$$

where $U_m = (u_m, v_m)$,

$$(\Lambda_m U_m)(x) = \begin{pmatrix} v_m(x) \\ -m^2\omega^2 u_m(x) + \rho(x)u_m(x) \end{pmatrix}, \quad \text{and} \quad N_m(U) = \begin{pmatrix} 0 \\ -\gamma r(x) g_m(x) \end{pmatrix}.$$

3.1 The phase space

Fix $s \geq 0$. Then for fixed $x \in \mathbb{R}$ we look for solutions $U = (U_m)_{m\in\mathbb{N}_{odd}}$ in the space $\ell^1(s)$ equipped with the norm

$$\|U\|_{\ell^1(s)} = \sum_{m\in\mathbb{N}_{odd}} \|U_m\|_{\mathbb{C}^2} |m|^s$$

with the extension $U_{-m} = \overline{U_m}$ and $U_{2m} = 0$. The space is closed under convolution, i.e.

$$\|U * V\|_{\ell^1(s)} \leq \|U\|_{\ell^1(s)} \|V\|_{\ell^1(s)}, \quad \text{where} \quad (U * V)_m = \sum_{k\in\mathbb{Z}} U_{m-k} V_k.$$

As a consequence the nonlinearity $N = (N_m)_{m\in\mathbb{N}}$ is a smooth trilinear mapping from $\ell^1(s)$ to $\ell^1(s)$. An operator $\Lambda = (\Lambda_m)_{m\in\mathbb{N}}$ acting on U by

$$(\Lambda U)_m = \Lambda_m U_m$$

can be estimated by

$$\|\Lambda U\|_{\ell^1(s)} \leq (\sup_{m\in\mathbb{N}_{odd}} \|\Lambda_m\|_{\mathbb{C}^2\to\mathbb{C}^2}) \|U\|_{\ell^1(s)}. \tag{3.2}$$

If $U \in \ell^1(s)$ then $t \mapsto \sum_{k\in\mathbb{Z}} U_k e^{ik\omega t}$ is s-times continuously differentiable. Since every $s \geq 0$ is possible the constructed modulated pulse solutions will be arbitrarily smooth w.r.t. t.

3.2 The reversibility

The persistence of the approximate homoclinic solution w.r.t. higher order perturbations heavily relies on the reversibility of (3.1) which we write as

$$\dot{U} = \mathbf{F}(x, U). \tag{3.3}$$

Definition 3.1. *The non-autonomous system* (3.3) *is called reversible if there is a reversibility operator* $\mathbf{R}$ *such that*

$$\mathbf{R}\mathbf{F}(x, U) = -\mathbf{F}(-x, \mathbf{R}U).$$

Lemma 3.2. *With $x \mapsto U(x)$ solving* (3.3)*, also $x \mapsto V(x) = \mathbf{R}U(-x)$ is a solution.*

Proof. We have

$$\dot{V}(x) = -\mathbf{R}\dot{U}(-x) = -\mathbf{R}\mathbf{F}(-x, U(-x)) = \mathbf{F}(x, \mathbf{R}U(-x)) = \mathbf{F}(x, V(x)).$$

□

For system (3.1) we define a reversibility operator $\mathbf{R}$ by $\mathbf{R} = \oplus_{m \in \mathbb{N}_{odd}} R$ with

$$R(u_m, v_m) = (u_m, -v_m).$$

System (3.1) is reversible, i.e., invariant under $(x, u_m, v_m) \mapsto (-x, u_m, -v_m)$, due to the assumption that ρ and r are even functions. Lemma 3.2 then implies that with $x \mapsto U(x) = (u_m, v_m)_{m \in \mathbb{N}_{odd}}(x)$ a solution, also $x \mapsto \mathbf{R}U(-x) = (u_m, -v_m)_{m \in \mathbb{N}_{odd}}(-x)$ is a solution.

3.3 Preserving the reversibility

Due to the above theorem of Floquet (see Section 2) the solutions of $\partial_x U_m = \Lambda_m U_m$ are given by $U_m(x) = P_m(x) e^{xM_m} U_m(0)$ with $P_m(x) = P_m(x + 4\pi)$ and $M_m \in \mathbb{C}^{2\times 2}$. In order to make the linear part of the system autonomous we could make a change of variables $U_m(x) = P_m(x)V_m(x)$. However, this choice would make the use of the reversibility really awkward. Moreover, in general M_m is not in Jordan normal form. Therefore, we proceed as follows: First, we rewrite

$$\begin{aligned} U_m(x) &= P_m(x) e^{xM_m} U_m(0) \\ &= P_m(x) S_m^{-1} e^{xJ_m} S_m U_m(0) \\ &= Q_m(x) e^{xJ_m} V_m(0) \end{aligned}$$

with $Q_m(x) = P_m(x) S_m^{-1}$ such that $V_m(x)$ defined by

$$U_m(x) = Q_m(x) V_m(x) \tag{3.4}$$

satisfies $\partial_x V_m = J_m V_m$ where J_m is the Jordan normal form of M_m and S_m the associated transformation. According to the explanations in Section 2.3 we have to distinguish two cases, namely two Floquet exponents which are not the same, if $\lambda_{1,2} \in \Sigma_3^e \cup \Sigma_4^e$, and a Jordan block to the Floquet exponent 0, if $\lambda_{1,2} \in \Sigma_2^e$.

Case I: Assume first that the Floquet exponents for fixed m satisfy $\lambda_1 \neq \lambda_2$. The solutions of $\partial_x U_m = \Lambda_m U_m$ can be written as

$$U(x) = c_1 \psi_1(x) + c_2 \psi_2(x) = c_1 e^{\lambda_1 x} \phi_1(x) + c_2 e^{\lambda_2 x} \phi_2(x)$$

with constants c_j and 4π-periodic ϕ_j here and in the following. Since the systems are reversible with $x \mapsto e^{\lambda_1 x} \phi_1(x)$ also $x \mapsto e^{-\lambda_1 x} R\phi_1(-x)$ is a solution. Hence we define the second fundamental solution

$$e^{\lambda_2 x} \phi_2(x) = e^{-\lambda_1 x} R\phi_1(-x),$$

which implies $\lambda_2 = -\lambda_1$ and $\phi_2(x) = R\phi_1(-x)$. We introduce the new variable $V(x) = (v_1, v_2)(x)$ by

$$U(x) = v_1(x)\phi_1(x) + v_2(x)\phi_2(x) = (\phi_1(x), \phi_2(x)) \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$$

where by construction $\partial_x V(x) = BV(x)$ with $B = \mathrm{diag}(\lambda_1, \lambda_2)$. Hence, the above change of variables (3.4) and the last change of variables coincide, i.e. $B = J_m$, and the linear system is now reversible w.r.t. the transformed reversibility operator $\tilde{R}_m$ defined through

$$\tilde{R}_m \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}.$$

Case II: Next assume that we have a Jordan-block at $\lambda = 0$. Then

$$U(x) = c_1\psi_1(x) + c_2\psi_2(x) = c_1\phi_1(x) + c_2(x\phi_1(x) + \phi_2(x)).$$

Due to the reversibility $\phi_1(x) = R\phi_1(-x)$, and $\phi_2(x) = -R\phi_2(-x)$. We introduce the new variable $V(x) = (v_1, v_2)(x)$ by

$$U(x) = v_1(x)\phi_1(x) + v_2(x)\phi_2(x) = (\phi_1(x), \phi_2(x)) \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$$

where by construction $\partial_x V(x) = BV(x)$, with $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case the representation of the reversibility operator is preserved, i.e.

$$\tilde{R}_m \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}.$$

Remark 3.3. *By the 4π-periodic setting we avoid Jordan-blocks with eigenvalue* $\mathrm{i}/2$. *For such Jordan-blocks the new representation $\tilde{R}_m$ of the reversibility operator then depends on x which would not be very useful for the subsequent analysis. The x-dependency of $\tilde{R}_m$ could also be avoided by doubling these coordinates but using Jordan-blocks with the equivalent (in term's of Floquet's theorem) eigenvalue* $-\mathrm{i}/2$. *However, the overall construction would be much more complicated than the chosen one.*

3.4 Conjugation of the old and the new reversibility operator

The old reversibility operator $\mathbf{R}$ and the new reversibility operator $\tilde{\mathbf{R}} = \oplus_{m\in\mathbb{N}_{odd}} \tilde{\mathbf{R}}_m$ are conjugated via the transform $U(x) = Q(x)V(x)$ where $Q(x) = \oplus_{n_j\in\mathbb{N}_{odd}} Q_m(x)$. We find

$$Q(-x)\tilde{\mathbf{R}} = \mathbf{R}Q(x)$$

which implies $Q^{-1}(-x)\mathbf{R} = \tilde{\mathbf{R}}Q^{-1}(x)$. By the analysis of the last subsection we already know that the transformed linear operator is reversible w.r.t. to the new reversibility operator $\tilde{\mathbf{R}}$.

Lemma 3.4. *The transformed nonlinearity $\tilde{N}(x,V) = Q^{-1}(x)N(x,Q(x)V)$ is reversible w.r.t. to the new reversibility operator $\tilde{\mathbf{R}}$, i.e., we have*

$$\tilde{\mathbf{R}}\tilde{N}(x,V) = -\tilde{N}(-x,\tilde{\mathbf{R}}V).$$

Proof. This holds according to

$$\begin{aligned}\tilde{\mathbf{R}}\tilde{N}(x,V) &= \tilde{\mathbf{R}}Q^{-1}(x)N(x,Q(x)V) = Q^{-1}(-x)\mathbf{R}N(x,Q(x)V)\\ &= -Q^{-1}(-x)N(-x,\mathbf{R}Q(x)V) = -Q^{-1}(-x)N(-x,Q(-x)\tilde{\mathbf{R}}V)\\ &= -\tilde{N}(-x,\tilde{\mathbf{R}}V).\end{aligned}$$

□

3.5 The reversible change of variables

With $U_m(x) = Q_m(x)V_m(x)$ we find

$$\partial_x V_m(x) = B_m V_m(x) + \tilde{N}_m(V), \quad m \in \mathbb{N}_{odd}, \tag{3.5}$$

with

$$\tilde{N}_m(V) = (Q_m(x))^{-1}N_m(x,(Q_j(x)V_j(x))_{j\in\mathbb{N}_{odd}}).$$

For $\varepsilon = 0$ we have by assumption that B_1 is a Jordan block of size 2 with associated eigenvalue 0. All other B_m with $m \geq 3$ possess either one positive and one negative eigenvalue or two conjugated purely imaginary eigenvalues or a Jordan block with eigenvalue 0. Due to the closing of the spectral gaps for $m \to \infty$ the eigenvalues are on or converge towards the imaginary axis for $m \to \infty$.

By our construction System (3.5) is reversible w.r.t. the transformed reversibility operator $\tilde{\mathbf{R}}$.

The change of variables is bounded in the following sense.

Lemma 3.5. *Let $Q_m(x) = \begin{pmatrix} q_{11,m}(x) & q_{12,m}(x) \\ q_{21,m}(x) & q_{22,m}(x)\end{pmatrix}$. Then there exists a $C > 0$ such that*

$$\sup_{m\in\mathbb{N}_{odd}}\sup_{x\in[0,4\pi]}(|q_{11,m}(x)| + |q_{12,m}(x)|) < C \text{ and } \sup_{x\in[0,4\pi]}\|(Q_m(x))^{-1}\|_{\mathbb{C}^2\to\mathbb{C}^2} < C.$$

Proof. See Appendix C. □

Remark 3.6. *Although a periodic step function ρ does not satisfy $\rho \in H^1_{\mathrm{per}}$ the validity of this lemma can be checked explicitly for such ρ.*

Corollary 3.7. *The transformed nonlinearity $\tilde{N}$ is a smooth trilinear mapping from $\ell^1(s)$ to $\ell^1(s)$.*

Proof. Since in N_m only the first coordinate of the U_m occurs, after the transforms $U_m = Q_m V_m$ only $q_{11,m}$ and $q_{12,m}$ occur in the transformed nonlinearity. The result follows then from Lemma 3.5 which gives bounds for $q_{11,m}$, $q_{12,m}$ and Q_m^{-1}, the estimate (3.2), and the fact that $\ell^1(s)$ is closed under convolutions. □

4 The normal form transform

The next step is to rewrite the infinite dimensional system (3.5) in the form (1.8)-(1.9) We set $q = V_1$ and $w = (V_3, V_5, \ldots)$. Then (q, w) satisfy

$$\partial_x q = B_1 q + F(q, w)\,, \tag{4.1}$$

$$\partial_x w = Lw + G(q, w) + H(q)\,, \tag{4.2}$$

with $L = \mathrm{diag}(B_3, B_5, \ldots)$, and

$$\begin{aligned} F(q, w) &= \tilde{N}_1, \\ G(q, w) + H(q) &= (\tilde{N}_3, \tilde{N}_5, \ldots), \end{aligned}$$

where G is chosen such that $G(q, 0) = 0$. In the following we do not distinguish between v_1 and $(v_1, 0, \ldots)$, etc..

We proceed as described in the introduction. By a number of normal form transforms the term $H(q)$ can be made small in terms of q. Since G depends at least linearly upon w we expect that the lowest order of H in terms of q gives the size of w. The smallest possible order w.r.t q in H depends on the validity of some non resonance condition. We first assume that we have the situation described in Theorem 1.1, i.e. we make the following assumption.

Assumption 4.1. *For $\varepsilon = 0$ there exist $N \in \mathbb{N}$ such that $\lambda_j \notin \mathrm{i}\mathbb{Z}$ for $3 \leq j < N$.*

This is the assumption for a 2π-periodic system and ω_0 touching a band edge at $l = 0$. The 4π-periodic case and that ω_0 touches a band edge at $l = 1/2$ is handled subsequently.

Remark 4.2. *It is well known that in autonomous systems in diagonalized form, a term $V_1^m V_{-1}^n$ can be eliminated in the equation for V_j if the non-resonance condition*

$$\lambda_j - m\lambda_1 - n\lambda_{-1} \neq 0$$

is satisfied, cf. [13]. The condition for time-periodic systems is different from the one for autonomous systems (see the end of the proof of the next lemma), namely

$$\lambda_j - m\lambda_1 - n\lambda_{-1} \notin \mathrm{i}\mathbb{Z}.$$

Since $\lambda_1 = \lambda_{-1} = 0$ for $\varepsilon = 0$ this condition reduces in our case to $\lambda_j \notin \mathrm{i}\mathbb{Z}$. Due to $\left(e^{\mathrm{i}\omega t}\right)^3 = e^{3\mathrm{i}\omega t}$, terms of order $\mathcal{O}(\|q\|^3)$ only occur in the equation for U_3 and U_{-3}. Hence the condition for eliminating the terms of order $\mathcal{O}(\|q\|^3)$ in H is $\lambda_3, \lambda_{-3} \notin \mathrm{i}\mathbb{Z}$. Similarly removing the terms of order $\mathcal{O}(\|q\|^5)$ in H requires $\lambda_3, \lambda_{-3}, \lambda_5, \lambda_{-5} \notin \mathrm{i}\mathbb{Z}$. Thus, if Assumption 4.1 is valid all terms of order $\mathcal{O}(\|q\|^n)$ with $3 \leq n < N$ can be eliminated in the equation for w by a finite-dimensional transformation.

We prove

Lemma 4.3. *Under the validity of Assumption 4.1 there is a 2π-periodic, near-identity and finite-dimensional change of variables $\tilde{w} = \tilde{\Phi}(q,w)$ in a neighborhood of $(q,w,\varepsilon) = (0,0,0)$ which transforms the system* (4.1)-(4.2) *into*

$$\partial_x q = B_1 q + \tilde{F}(q,\tilde{w})\,, \tag{4.3}$$

$$\partial_x \tilde{w} = L\tilde{w} + \tilde{G}(q,\tilde{w}) + \tilde{H}(q) \tag{4.4}$$

where the nonlinear term $\tilde{H}$ is of order $\mathcal{O}(\|q\|^N)$ and where $\tilde{G}(q,0) = 0$. Moreover, $\tilde{F},\tilde{G} : \ell^1(s)\times\ell^1(s) \to \ell^1(s)$, and $\tilde{H} : \ell^1(s) \to \ell^1(s)$ are smooth mappings.

Proof. The statement is proved by induction. First the system (4.1)-(4.2) is completed with $\partial_x\varepsilon = 0$ such that all terms with an ε can be handled as nonlinear. Let us assume

$$\|H(q)\| = \mathcal{O}(\|q\|^p)$$

for a $p \in \mathbb{N}$ with $p < N$. We construct a 2π-periodic near identity change of variables which removes the homogeneous terms of degree p w.r.t. q in H. As explained in Remark 4.2 $H(q)$ has only components in the equations for w_k with $3 \le k \le p$.

We make the finite dimensional change of variables

$$\tilde{w}_k = w_k + J_k(x, \underbrace{q,\ldots,q}_{p}), \quad J_k \ : \ p-\text{linear w.r.t. } q, \quad 3 \le k \le p,$$

$$\tilde{w}_k = w_k, \quad k > p.$$

Differentiating $\tilde{w}_k$ w.r.t. x yields for $3 \le k \le p$ that

$$\tilde{w}_k' = w_k' + J_k(x,q)' = B_k w_k + \ G_k(x,q,w) + H_k(x,q) + J_k(x,q)'$$

where we used the abbreviation $\cdot' = (d/dx)\cdot$. Then expressing w_k in terms of $\tilde{w}_k$ gives

$$\tilde{w}_k' = B_k(\tilde{w}_k - J_k(x,q)) + G_k(x,q,\tilde{w} - J(x,q)) + H_k(x,q) + J_k(x,q)'$$

where

$$J_k(x,q)' = \partial_x J_k(x,q) + \sum_{j\le p} J_k(x,q,\ldots,\underbrace{B_1 q}_{\text{pos. } j},\ldots,q) + h.o.t..$$

The higher order terms $h.o.t.$ contain the terms $J_k(x,q,\ldots,\underbrace{F}_{\text{pos. } j},\ldots,q)$ which are at least of order $p+2$ w.r.t. q and thus $h.o.t.$ contains no term to be removed in this step.

In order to eliminate the homogeneous terms $H_{k,\text{hom}}$ of degree p of $H(q)$ we have to choose $J_k(x,q)$ to satisfy

$$B_k J_k(x,q) = \partial_x J_k(x,q) + \sum_{j=0}^{p} J_k(x,q,\ldots,\underbrace{B_1 q}_{\text{pos. } j},\ldots,q) + H_{k,\text{hom}}(x,q). \tag{4.5}$$

The term $H_{k,\text{hom}}$ is of the form

$$H_{k,\text{hom}}(x,q) = \begin{pmatrix} H_{k1}(x,q) \\ H_{k2}(x,q) \end{pmatrix}, \quad H_{kj}(x,q) = \sum_{n=0}^{p} h_{jn}(x) y_n(q)$$

where $y_n(q) = q_2^n q_1^{p-n}$ is a homogeneous polynomial in q of order p. Since q is two-dimensional there are $p+1$ of them. Hence we choose J_k to be of a similar form, namely

$$J_k(x,q) = \begin{pmatrix} j_{1k}(x,q) \\ j_{2k}(x,q) \end{pmatrix}, \quad j_{lk}(x,q) = \sum_{n=0}^{p} q_{ln}(x) y_n(q), \quad l \in \{1,2\}$$

For $\varepsilon = 0$ we have $B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so that

$$y_n(q, \dots, \underbrace{B_1 q}_{\text{pos. } j}, \dots, q) = q_2 \cdot \ldots \cdot (B_1 q)_2 \cdot \ldots \cdot q_2 \cdot q_1 \cdot \ldots \cdot q_1 = 0$$

if $(B_1 q)_2 = 0$ is is involved, i.e. for $j \leq n$ and

$$y_n(q, \dots, \underbrace{B_1 q}_{\text{pos. } j}, \dots, q) = q_2 \cdot \ldots \cdot q_2 \cdot q_1 \cdot \ldots \cdot (B_1 q)_1 \cdot \ldots \cdot q_1 = y_{n+1}(q)$$

if $(B_1 q)_1 = q_2$ is involved, i.e., for $j > n$. According to the non-resonance condition the B_k for $k \geq 3$ are diagonal with eigenvalues $\pm\lambda_k$ or a Jordan-block with eigenvalue $\pm i/2$. Equation (4.5) transforms in the first case into

$$\begin{aligned} \lambda_k \, q_{1n} &= q_{1n}' + (p-n) q_{1(n-1)} + h_{1n}, \\ -\lambda_k \, q_{2n} &= q_{2n}' + (p-n) q_{2(n-1)} + h_{2n}. \end{aligned}$$

for $0 \leq n \leq p$ with the convention that $q_{l(-1)} = 0$. Since we have 2π-periodic coefficients h_{ln} we use Fourier series $f = \sum_m \hat{f}_m e^{imx}$ and obtain

$$\begin{aligned} \lambda_k \, \hat{q}_{1n,m} &= \mathrm{i}m \, \hat{q}_{1n,m} + (p-n) \hat{q}_{1(n-1),m} + \hat{h}_{1n,m} \\ -\lambda_k \, \hat{q}_{2n,m} &= \mathrm{i}m \, \hat{q}_{2n,m} + (p-n) \hat{q}_{2(n-1),m} + \hat{h}_{2n,m} \end{aligned}$$

which is solved recursively starting at $n = 0$ by

$$\hat{q}_{1n,m} = \frac{\hat{h}_{1n,m} + (p-n)\hat{q}_{1(n-1),m}}{\lambda_k - im}, \quad \hat{q}_{2n,m} = \frac{\hat{h}_{2n,m} + (p-n)\hat{q}_{2(n-1),m}}{-\lambda_k - im} \tag{4.6}$$

For $k \leq p < N$ the denominator never vanishes according to Assumption (4.1). The second case, i.e. B_k a Jordan-block, can be handled similarly and we find the same conditions. See also Appendix A where an example is considered. Therefore, we are done. □

In the 4π-periodic situation the non-resonance condition $\lambda_j \notin \mathrm{i}\mathbb{Z}$ for $l_0 = 0$ obviously has to be replaced by $2\lambda_j \notin \mathrm{i}\mathbb{Z}$.

The additional resonances which occur by doubling the period come from the eigenvalues $\lambda_j = \pm \mathrm{i}/2$ of the 2π-periodic case. In the 4π-periodic case they correspond to $\lambda_j = 0$. The eigenfunctions to these new zero eigenvalues are odd in the 4π-periodic setting, whereas the eigenfunctions of the original zero eigenvalues are even in the 4π-periodic setting. Hence in case of a vanishing denominator in (4.6) also the nominator, which is the projection of an even function on an odd function vanishes. Hence, we get back the original non-resonance condition $\lambda_j \notin \mathrm{i}\mathbb{Z}$.

A similar argument holds in case $l_0 = 1/2$. In this case an odd power of an odd function is projected on an even function which also gives zero.

5 Construction of a reversible homoclinic solution in case $H = 0$

The final goal of the normal form transforms would be to get a finite-dimensional system for q alone. As explained in the introduction this goal cannot be reached. After applying the normal form transforms we still have a system of the form

$$\partial_x q = B_1 q + F(q, w), \tag{5.1}$$

$$\partial_x w = Lw + G(q, w) + H(q), \tag{5.2}$$

but now with $H = \mathcal{O}(\|q\|^N)$. Since G is at least linear w.r.t. w we expect that the lowest order of H in terms of q gives the size of w. It turns out that for the bifurcating solutions $w = \mathcal{O}(\|q\|^{N-1})$ at least for a large interval w.r.t. x.

The idea is now to consider H as a small perturbation. In case $H = 0$ the set $\{w = 0\}$ is an invariant subspace and we have a two-dimensional reduced system

$$\partial_x q(x) = B_1 q(x) + F(x, q(x), 0), \tag{5.3}$$

but still with 4π-periodic x-dependent coefficients. By another normal form transform and truncation a reversible x-independent system can be derived for which the existence of two homoclinic orbits can be established by phase plane analysis. These homoclinic solutions will not persist as homoclinic solutions to the origin if $H \neq 0$, but they will be the basis of our subsequent analysis.

In a first step in (5.3) for the homogeneous polynomials of degree $< N$ w.r.t q we transfer the x-dependent coefficients in x-independent ones.

Lemma 5.1. *There is a reversible 4π-periodic near-identity change of variables which transforms* (5.3) *into*

$$\partial_x q(x) = B_1 q(x) + F_1(q(x), 0) + F_2(x, q(x), 0), \tag{5.4}$$

with $F_2 = \mathcal{O}(\|q\|^N)$ and F_1 not depending explicitly on x. Moreover, $F_1 : \ell^1(s) \times \ell^1(s) \to \ell^1(s)$ and $F_2 : \ell^1(s) \times \ell^1(s) \to \ell^1(s)$ can be extended to smooth mappings w.r.t. q and w by leaving the terms with a w unchanged.

Proof. The proof goes along the lines of the one for Lemma 4.3. The major difference to this proof is that in the denominator of the right hand sides of (4.6) now $\lambda_k = 0$. Hence, the zeroth Fourier coefficient cannot be removed, i.e., the x-independent part remains. $\square$

We will handle F_2 like H as a small perturbation. Hence, our final reduced system is given by

$$\partial_x q = B_1 q + F_1(q, 0). \tag{5.5}$$

Since B_1 is a Jordan-block of size two for $q = (a, b)$ System (5.5) has the form

$$\begin{aligned} \partial_x a &= b + \mathcal{O}(|\varepsilon^2 a|, |\varepsilon^2 b|, |a^3|, \ldots, |b^3|), \\ \partial_x b &= \mathcal{O}(|\varepsilon^2 a|, |\varepsilon^2 b|, |a^3|, \ldots, |b^3|) \end{aligned}$$

for small (a, b). Introducing A and B by

$$a(x) = \varepsilon A(\varepsilon x) \quad \text{and} \quad b(x) = \varepsilon^2 B(\varepsilon x)$$

yields

$$\partial_X A = B + \mathcal{O}(\varepsilon), \qquad \partial_X B = s_1 A + \gamma s_3 A^3 + \mathcal{O}(\varepsilon), \tag{5.6}$$

with constants $s_1, s_3 \in \mathbb{R}$, where $s_1 \sim |\varepsilon^{-2}(\omega^2 - \omega_{n_0}^2)| = \mathcal{O}(1) > 0$. If $\gamma s_3 < 0$ the truncated system

$$\partial_X A = B, \qquad \partial_X B = s_1 A + \gamma s_3 A^3, \tag{5.7}$$

possesses a pair of homoclinic solutions $Q_{hom} = (A_{hom}, B_{hom})$ which is given by

$$A_{hom}(X) = \pm\sqrt{\frac{2s_1}{-\gamma s_3}}\,\text{sech}\left(\sqrt{s_1}X\right), \qquad \partial_X A_{hom} = B_{hom}.$$

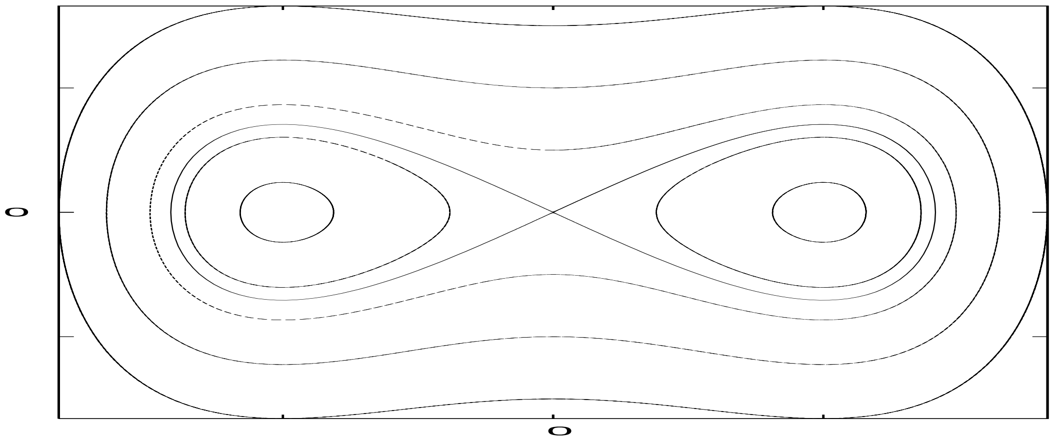

Figure 9: The phase portrait of the reduced system in the (A, B)-plane with the homoclinic orbits.

The persistence of these homoclinic solutions which lie in the intersection of the stable and unstable manifold of the origin under the perturbations of order $\mathcal{O}(\varepsilon)$ follows with the usual reversibility argument, namely the transversal intersection of the homoclinic orbit with the fixed space of reversibility $\{B = 0\}$ for $\varepsilon = 0$. By the transversal intersection for $\varepsilon = 0$ the unstable manifold intersects the fixed space of reversibility also for $\varepsilon > 0$. With the solution $(A, B) : (-\infty, 0] \to \mathbb{R}^2$ satisfying $B(0) = 0$, also $(\tilde{A}, \tilde{B}) : [0, \infty) \to \mathbb{R}^2$ with $(\tilde{A}, \tilde{B})(X) = (A, -B)(-X)$ is a solution. Hence, by gluing together these two solutions at $(A(0), 0)$ we have established the existence of a pair of homoclinic solutions for (5.5). See Figure 10.

6 Counting the dimensions

Although the homoclinic solutions $q^{hom} = \varepsilon Q_{hom}$ constructed in the last section will not persist if $H \neq 0$, they will be the basis of our subsequent analysis for the full system

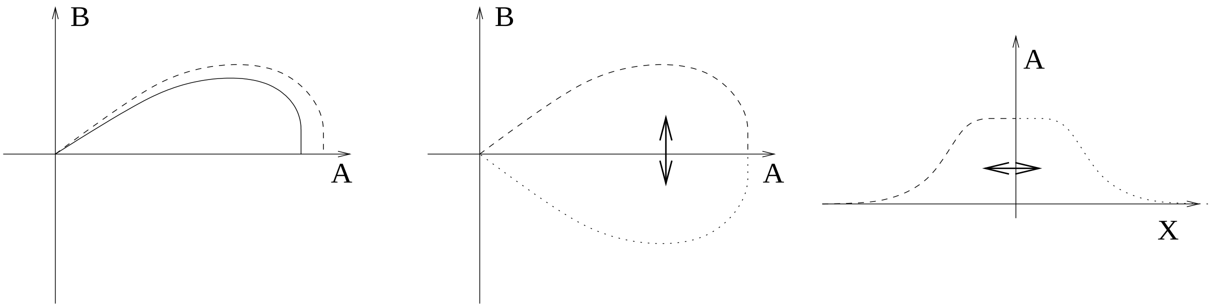

Figure 10: The first panel shows the transversal intersection of the unstable manifold (solid line) with the fixed space of reversibility $\{B = 0\}$ for the lowest order system (5.7). The dotted line is the unstable manifold intersecting the fixed space of reversibility also transversally in the full system (5.5). The next panel shows the reversible reflection of this manifold and the last panel the associated symmetric solution.

which we write as

$$\partial_x q = B_1 q + F_1(q, 0) + \delta(F_1(q, w) - F_1(q, 0) + F_2(q, w)), \tag{6.1}$$

$$\partial_x w = Lw + \delta(G(q, w) + H(q)), \tag{6.2}$$

where the perturbation terms are indicated with δ. The unperturbed system is given for $\delta = 0$ and the perturbed system is given for $\delta = 1$.

In order to prove the persistence of the bifurcating homoclinic solutions which have been found for $H = 0$, also in case $H \neq 0$, when $\{w = 0\}$ is no longer an invariant subspace, we can proceed as in previous section, i.e. we can use reversibility arguments. The reversibility argument for the persistence of homoclinic solutions in general is based on a transversal intersection of the one-dimensional stable manifold associated to the eigenvalue leaving zero in the direction of the negative real axis with the fixed space of reversibility $\{(u, \partial_x u) = (u, 0) \mid u : [0, 2\pi/\omega) \to \mathbb{R}\}$ of (2.1). Obviously, many dimensions are missing for a transversal intersection. However, the center-stable manifold (which also includes the w-modes) and the fixed space of reversibility intersect transversally.

Lemma 6.1. *For $\delta = 0$ the fixed space of reversibility and the center-stable manifold intersect transversally, i.e. span the complete phase space $\ell^1(s)$ at the intersection point.*

Proof. For the subspace to $m = 1$, i.e. for q, we have the transversal intersection by the analysis of the last section. For all other m it is sufficient to prove that the (restricted) fixed space of reversibility and the (restricted) center stable subspace at $w = 0$ span $\mathbb{R}^2$. For the m-th subspace the fixed space of reversibility in case of Floquet exponents $\lambda_1 \neq \lambda_2$ is given by $\{\mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \mu \in \mathbb{R}\}$ and in case of a Jordan-block with eigenvalue zero by $\{\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \mu \in \mathbb{R}\}$.

In case of $\lambda_1, \lambda_2 \in \mathrm{i}\mathbb{R}$ the center-stable manifold part is all of $\mathbb{R}^2$. Hence we have a transversal intersection in the diagonal and Jordan-block case. In the remaining case $\lambda_1 < 0 < \lambda_2$ the center-stable part is given by $\{\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \mu \in \mathbb{R}\}$ which intersects the fixed space of reversibility which is given in this case by $\{\mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \mu \in \mathbb{R}\}$ transversally. Hence the direct sums of the fixed spaces of reversibility and the direct sums of the center-stable subspaces together span the complete phase space $\ell^1(s)$. □

For $\delta = 1$ the transversality of the center-stable manifold and of the fixed space of reversibility follows with some perturbation argument using the subsequent estimates.

7 Estimates for the solutions on the center-stable manifold

In the last section we have seen that there is a transversal intersection of the fixed space of reversibility and the center stable-manifold for $\delta = 0$. By a perturbation argument we can prove that there is a transversal intersection also for $\delta = 1$. By our construction with the reversible reflection the solutions we are interested in lie in the center-stable manifold for $x \in [0, \infty)$ and in the center-unstable manifold for $x \in (-\infty, 0]$. For $x \to \infty$ solutions on the center-stable manifold converge towards the center manifold. Hence for $x \to \infty$ polynomial growth of these modes is possible and so estimates for all $x \in \mathbb{R}$ cannot be expected. However, in case $H(q) = \mathcal{O}(\|q\|)^N$ for an $N \in \mathbb{N}$ fixed it can be shown that $w = \mathcal{O}(\varepsilon^{N-1})$ for a very large interval, first for all $x \in [0, \varepsilon^{3-2N}]$ and then by the reversible reflection in the next section for all $x \in [-\varepsilon^{3-2N}, \varepsilon^{3-2N}]$. Moreover, q will be of homoclinic form and $\mathcal{O}(\varepsilon^{N-1})$-close to $q^{hom} = \varepsilon Q^{hom}$.

In order to prove such bounds for w and q we make the ansatz

$$(q, w) = (q^{hom} + \varepsilon^{N-1}\mathcal{R}_1, \varepsilon^{N-1}\tilde{\mathcal{R}}).$$

The deviation $\mathcal{R} = (\mathcal{R}_1, \tilde{\mathcal{R}})$ from the homoclinic orbit in the transformed systems (6.1) and (6.2) satisfies

$$\left\{ \begin{array}{l} \partial_x \mathcal{R}_1 = B_1 \mathcal{R}_1 + \check{N}_1(\mathcal{R}), \\ \partial_x \tilde{\mathcal{R}} = L\tilde{\mathcal{R}} + \check{N}(\mathcal{R}), \end{array} \right. \tag{7.1}$$

where

$$\check{N}_1(\mathcal{R}) = \varepsilon^{1-N}\Big((F_1 + F_2)(\varepsilon Q^{hom} + \varepsilon^{N-1}\mathcal{R}_1, \varepsilon^{N-1}\tilde{\mathcal{R}}) - F_1(\varepsilon Q^{hom}, 0)\Big),$$
$$\check{N}(\mathcal{R}) = \varepsilon^{1-N}\Big(G(\varepsilon Q^{hom} + \varepsilon^{N-1}\mathcal{R}_1, \varepsilon^{N-1}\tilde{\mathcal{R}}) + H(\varepsilon Q^{hom} + \varepsilon^{N-1}\mathcal{R}_1)\Big).$$

We define projections $P_{1,s}, P_{1,u}$ on the stable and unstable eigenspaces $E_{1,s}$, $E_{1,u}$ and $(P_{m,cs}, P_{m,u})_{m>1}$, on the center-stable and unstable eigenspaces $\tilde{E}_{cs}$, $\tilde{E}_u$ associated to all eigenvalues with $\mathrm{Re}\lambda_m \leq 0$ and $\mathrm{Re}\lambda_m > 0$ respectively.

The projections are uniformly bounded and well-defined since they are defined componentwise for $B_m \in \mathbb{C}^{2\times 2}$. For the construction of a center-stable manifold the nonlinearity has to be multiplied with some cut-off function θ being C_0^∞ with values in $[0, 1]$, satisfying $\theta(\rho) = 1$ for $\rho \leq 1$, and $\theta(\rho) = 0$ for $\rho \geq 2$. For a $C_\rho \geq 0$ sufficiently large, but fixed, independent of $0 \leq \varepsilon \ll 1$, we define a new nonlinearity by multiplying $\check{N}_1(\mathcal{R}), \check{N}(\mathcal{R})$ with $\theta(\|\mathcal{R}\|_{\ell^1(s)}/C_\rho)$. We denote the truncated nonlinearity with $\check{N}_1^\theta(\mathcal{R}), \check{N}^\theta(\mathcal{R})$.

We define

$$\Phi = (\Phi_{1,s}, \Phi_{1,u}, (\Phi_{m,cs}, \Phi_{m,u})_{m\in\mathbb{N}})$$

by

$$\left\{\begin{array}{rcl} \Phi_{1,s}(x) &=& e^{B_1 x} P_{1,s}\mathcal{R}_1(0) \\ && + \displaystyle\int_0^x e^{B_1(x-\xi)} P_{1,s} \check{N}_1^\theta((\mathcal{R}_j(\xi))_{j\in\mathbb{N}}) d\xi, \\ \Phi_{1,u}(x) &=& - \displaystyle\int_x^{1/\varepsilon^{2N-3}} e^{B_1(x-\xi)} P_{1,u} \check{N}_1^\theta((\mathcal{R}_j(\xi))_{j\in\mathbb{N}}) d\xi, \\ \Phi_{m,cs}(x) &=& e^{B_m x} P_{m,cs}\mathcal{R}_m(0) \\ && + \displaystyle\int_0^x e^{B_m(x-\xi)} P_{m,cs} \check{N}_m^\theta((\mathcal{R}_j(\xi))_{j\in\mathbb{N}}) d\xi, \\ \Phi_{m,u}(x) &=& - \displaystyle\int_x^{1/\varepsilon^{2N-3}} e^{B_m(x-\xi)} P_{m,u} \check{N}_m^\theta((\mathcal{R}_j(\xi))_{j\in\mathbb{N}}) d\xi. \end{array}\right. \tag{7.2}$$

Since we have a non-autonomous system the center-stable manifold depends on the starting time x_0. However, since our system is only reversible for starting time $x_0 = 0$ we restricted ourselves to starting time $x_0 = 0$ here and also in the previous sections.

The center-stable manifold is constructed by solving the equation $\mathcal{R} = \Phi(\mathcal{R})$ with a fixed point argument in the space $Y(1/\varepsilon^{2N-3})$ where

$$Y(x) = \{\mathcal{Y} \in C^0([0,x], \ell^1(s)) \mid \sup_{\xi\in[0,x]} \|\mathcal{Y}(\xi)\|_{\ell^1(s)} < \infty\}.$$

Since there exists a $C > 0$ such that for all $m \in \mathbb{N}_{\text{odd}}$

$$\left\{\begin{array}{rcll} \|e^{B_1 x} P_{1,s}\|_{\mathbb{C}^2\to\mathbb{C}^2} &\leq& Ce^{-\varepsilon x/2}, & \forall x \geq 0, \\ \|e^{B_1 x} P_{1,u}\|_{\mathbb{C}^2\to\mathbb{C}^2} &\leq& Ce^{-\varepsilon |x|/2}, & \forall x \leq 0, \\ \|e^{B_m x} P_{m,cs}\|_{\mathbb{C}^2\to\mathbb{C}^2} &\leq& C, & \forall x \geq 0, \\ \|e^{B_m x} P_{m,u}\|_{\mathbb{C}^2\to\mathbb{C}^2} &\leq& C, & \forall x \leq 0, \end{array}\right. \tag{7.3}$$

we find with the estimate (3.2) that

$$\|(e^{B_m x} P_{m,cs})_{m\in\mathbb{N}_{\text{odd}}}\|_{\ell^1(s)\to\ell^1(s)} \leq C, \qquad \forall x \geq 0,$$

and

$$\|(e^{B_m x} P_{m,u})_{m\in\mathbb{N}_{\text{odd}}}\|_{\ell^1(s)\to\ell^1(s)} \leq C, \qquad \forall x \leq 0.$$

Lemma 7.1. *The quantities $S_{1,s}(x) = \sup_{\xi\in[0,x]} \|\mathcal{R}_{1,s}(\xi)\|_{\mathbb{R}^2}, \ldots,$ and $S(x) = \|\mathcal{R}\|_{Y(x)}$ satisfy the inequalities*

$$\left\{\begin{array}{rcl} S_{1,s}(x) &\leq& C_{init} + C\varepsilon S(x) + C_{pulse}, \\ S_{1,u}(x) &\leq& C\varepsilon S(x) + C_{pulse}, \\ S_{m,cs}(x) &\leq& C_{init} + C(\varepsilon + \varepsilon^{2N-2}x)S(x) + C_{pulse}, \\ S_{m,u}(x) &\leq& C(\varepsilon + \varepsilon^{2N-2}x)S(x) + C_{pulse}. \end{array}\right. \tag{7.4}$$

The constant C_{init} is associated with $|\mathcal{R}_{1,s}(0)|, \|\mathcal{R}_{m,cs}(0)\|_{\ell^1(s)}$ and C_{pulse} with $|q(0)|$.

Proof. We have used the subsequent estimates:

1. Due to Corollary 3.7 and due to the use of the cut-off function we find

$$\begin{aligned} & \max(\|\check{N}_1^\theta(\mathcal{R})\|_{\mathbb{C}^2}, \|(\check{N}^\theta(\mathcal{R}))\|_{\ell^1(s)}) \\ \leq \; & C\Big(\varepsilon^2\|Q^{hom}\|_{\mathbb{C}^2}^2\|\mathcal{R}\|_{\ell^1(s)} + \varepsilon^N\|Q^{hom}\|_{\mathbb{C}^2}\|\mathcal{R}\|_{\ell^1(s)}^2 + \varepsilon^{2N-2}\|\mathcal{R}\|_{\ell^1(s)}^3 + \varepsilon\|Q^{hom}\|_{\mathbb{C}^2}^N\Big) \end{aligned}$$

   with C independent of $0 < \varepsilon \ll 1$.

2. Using (7.3) shows that we have for the first mode that

$$\left\|\int_0^x e^{B_1(x-\xi)}P_{1,s}f(\xi)d\xi\right\|_{\mathbb{C}^2} \leq C\varepsilon^{-1}\sup_{\xi\in[0,x]}\|f(\xi)\|_{\mathbb{C}^2}$$

   and

$$\left\|\int_x^{\varepsilon^{3-2N}} e^{B_1(x-\xi)}P_{1,u}f(\xi)d\xi\right\|_{\mathbb{C}^2} \leq C\varepsilon^{-1}\sup_{\xi\in[x,\varepsilon^{3-2N}]}\|f(\xi)\|_{\mathbb{C}^2}.$$

3. Since $\|Q^{hom}(\xi)\|_{\mathbb{C}^2} = \mathcal{O}(e^{-\varepsilon|\xi|})$ for $|\xi| \to \infty$ one has

$$\int_0^x \varepsilon\|Q^{hom}\|_{\mathbb{C}^2}^N d\xi \leq C,$$

$$\int_0^x \varepsilon^2\|Q^{hom}(\xi)\|_{\mathbb{C}^2}^2\|\mathcal{R}(\xi)\|_{\ell^1(s)}d\xi \leq C\varepsilon\|\mathcal{R}\|_{Y(x)},$$

$$\int_0^x \varepsilon^N\|Q^{hom}(\xi)\|_{\mathbb{C}^2}\|\mathcal{R}(\xi)\|_{\ell^1(s)}^2 d\xi \leq C\varepsilon^{N-1}\|\mathcal{R}\|_{Y(x)}^2.$$

4. Finally we have

$$\int_0^x \varepsilon^{2N-2}\|\mathcal{R}(\xi)\|_{\ell^1(s)}^3 d\xi \leq Cx\varepsilon^{2N-2}\|\mathcal{R}\|_{Y(x)}^3.$$

□

From (7.4) it follows for all $x \in [0, 1/\varepsilon^{2N-3}]$ and $\varepsilon > 0$ sufficiently small that

$$S(x) \leq 10(C_{init} + C_{\text{pulse}}).$$

This shows that Φ maps a ball with radius $10(C_{init} + C_{\text{pulse}})$ in $Y(1/\varepsilon^{2N-3})$ in itself. Similarly the contraction property of the map Φ for $\varepsilon > 0$ sufficiently small can be shown. Hence for sufficiently small $P_1\mathcal{R}_1(0)$ and $P_{m,cs}\mathcal{R}_m(0)$ there exists a fixed point $\mathcal{R}^* = \Phi(\mathcal{R}^*)$.

The **local center-stable manifold** $W_{cs}(\varepsilon)$ is a graph over the $P_1\mathcal{R}_1(0)$ and $P_{m,cs}\mathcal{R}_m(0)$ and is now defined as union of all points $\mathcal{R}^*(0)$ satisfying $\|\mathcal{R}^*\|_{Y(1/\varepsilon^{2N-3})} \leq 10(C_{init} + C_{\text{pulse}})$.

Exactly as in [11, Section 6], see also [10, 12], the transversal intersection of W_{cs} and the fixed space of reversibility follows. We refrain from repeating this proof and only make the following remark.

Remark 7.2. *In the proof of Lemma 6.1 in the two-dimensional subspaces the restrictions of the center stable manifold and of the fixed space of reversibility are all of $\mathbb{R}^2$ or have at least an angle of $\pi/4$ for $\delta = 0$. By the above estimates on $\mathcal{R}^*$ for $\delta = 1$ this picture is perturbed of order $\mathcal{O}(\varepsilon^{N-1})$. Hence the overall picture is not destroyed. However, the detailed proof would be much more involved.*

Remark 7.3. *In case that no Floquet exponents are on the imaginary axis and in case that they are uniformly bounded away from this axis the semigroups $e^{B_m x} P_{m,s}$ for $x \geq 0$ and $e^{B_m x} P_{m,u}$ for $x < 0$ show exponential decay rates. Then the estimates in* (7.4) *can be improved and no x occurs. Hence in this case the estimates can be proved for all $x \in \mathbb{R}$. The decay to zero for $|x| \to \infty$ can be shown by introducing some exponential weight. Hence in this case true breather solutions do exist. See [2].*

8 The reversible reflection

Now we have all ingredients to construct solutions for (3.1), resp. (3.5), resp. (6.1)-(6.2) which are of homoclinic form for all $x \in [-1/\varepsilon^{2N-3}, 1/\varepsilon^{2N-3}]$.

In the last section we constructed solutions (q, w) for (6.1)-(6.2) which are $\mathcal{O}(\varepsilon^{N-1})$-close for all $x \in [0, 1/\varepsilon^{2N-3}]$ to a true homoclinic solution constructed in Section 5 for the reduced system (5.5). Moreover, we know that the center-stable manifold where these solutions lie on intersects the fixed space of reversibility transversally, i.e. $(q, w)(0)$ are in the fixed space of the reversibility operator $\tilde{\mathbf{R}}$ defined in the subsections 3.2 and 3.3.

Moreover, by Lemma 3.4 we know that (3.5) is reversible in the sense of Definition 3.1. By Lemma 4.3 the same is true for (6.1)-(6.2). Hence by applying Lemma 3.2 to (6.1)-(6.2) we find solutions (q, w) for (6.1)-(6.2) which are $\mathcal{O}(\varepsilon^{N-1})$-close for all $x \in [-1/\varepsilon^{2N-3}, 0]$ to a true homoclinic solution constructed in Section 5 for the reduced system (5.5).

Since the solutions for $x \in [0, 1/\varepsilon^{2N-3}]$ and $x \in [-1/\varepsilon^{2N-3}, 0]$ have the point $(q, w)(0)$ in common we can glue these two parts together to find solutions (q, w) for (6.1)-(6.2) which are $\mathcal{O}(\varepsilon^{N-1})$-close for all $x \in [-1/\varepsilon^{2N-3}, 1/\varepsilon^{2N-3}]$ to a true homoclinic solutions constructed in Section 5 for the reduced system (5.5).

Undoing the transforms of subsection 3.3 and Lemma 4.3 concludes the proof of Theorem 1.1.

A An example

In this section we perform the calculations described in Sections 3–5 for a particular choice of the periodic coefficients ρ and r. Namely, we consider the equation

$$\partial_x^2 u = \partial_t^2 u + (\cos(x) + 2)u + u^3. \tag{A.1}$$

After casting (A.1) as an evolutionary system and taking its Fourier series, we arrive at a system of infinitely many ODEs

$$u_m'' = -m^2\omega^2 u_m + (\cos(x) + 2)u_m + g_m(u), \tag{A.2}$$

where $g_m(u)$ is defined in Equation (2.3) for $m \in \mathbb{Z}$. In this example we make the calculations for the first two modes, $m = 1$ and $m = 3$, since these modes have the largest contribution to the solution and calculations for higher modes are similar. The corresponding band gap structure can be found by solving the linear problem,

$$\partial_x^2 u_m = -m^2\omega^2 u_m + (\cos(x) + 2)u_m, \tag{A.3}$$

for various ω^2. We choose ω so $\omega^2 = \omega_{n_0}^2 - \varepsilon^2 = 1.6079$ where $\omega_{n_0}^2 \approx 1.6173$ is the first band edge and ε is chosen to be small enough so that ω falls in a gap. Here $\varepsilon^2 = 0.094$ is suitable. See Figure 11. We find $l_0 = 1/2$. By doubling the period from 2π to 4π this is transformed into $l_0 = 0$.

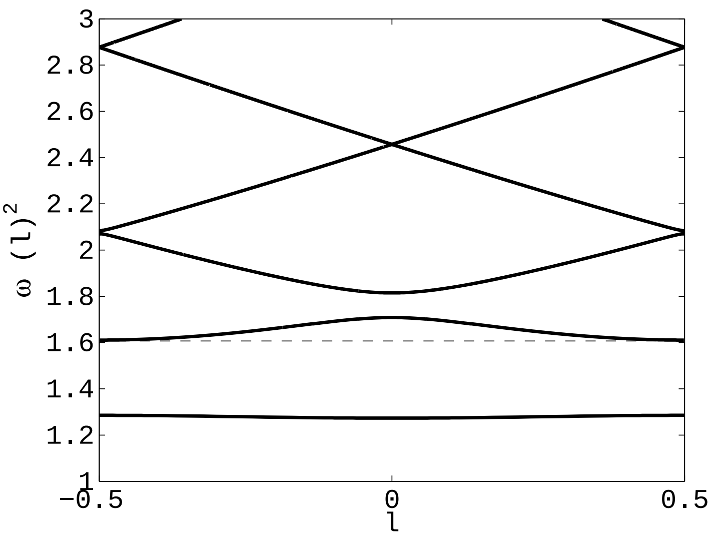

Figure 11: Eigenfunctions (solid lines) of the linear problem (A.3). A band edge (dashed line) was found at $\omega_{n_0}^2 \approx 1.6173$. We chose $\omega^2 = 1.6079$ so that it falls into the first spectral gap.

Using solutions of the linear problem, we are able to make a change of coordinates such that the linear problem is autonomous, see Section 3. After fixing $\omega^2 = 1.6079$ we can obtain the corresponding fundamental matrix to the first order system associated to (A.2) which has the form $\Phi_m(x) = P_m(x)e^{xM_m}$ with $P_m(x) = P_m(x + 4\pi)$ for each mode m. Since we left out in the proof of Lemma 4.3 the case of B_m being a Jordan-block we will perform here the calculations for tridiagonal matrices. Hence, we transform M_m with a matrix S_m in tridiagonal form. Moreover, for practical purposes in contrast to the proof of Lemma 4.3 the terms with ε will not be considered as nonlinear. As consequence B_1 below contains nonzero eigenvalues.

We set $B_m = S_m^{-1}M_mS_m$ and we define the new coordinate $U_m(x) = Q_m(x)V_m(x) = P_m(x)S_m^{-1}V_m$. In this example we find

$$B_1 = \begin{pmatrix} 0.0681 & 0.001 \\ 0.0 & -0.0681 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0.106\mathrm{i} & 0.230 \\ 0.0 & -0.106\mathrm{i} \end{pmatrix}. \tag{A.4}$$

See Figure 12 to see the corresponding $P_m(x)$ matrices.

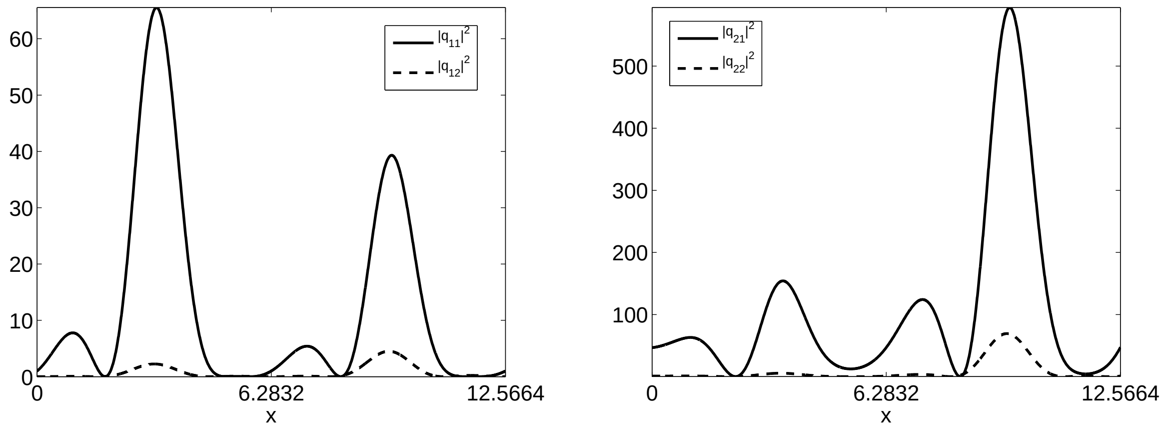

Figure 12: The 4π period functions corresponding to the elements of the matrix $Q_1(x)$ are shown. Those from the first row are in the left panel and those from the second are shown in the right panel.

We make use of the following notation

$$\begin{aligned} V_1(x) = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad V_3(x) &= \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}, \\ P_1(x)S_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad P_3(x)S_3 &= \begin{pmatrix} q_{11}^3 & q_{12}^3 \\ q_{21}^3 & q_{22}^3 \end{pmatrix}, \\ (P_1(x)S_1)^{-1} = \begin{pmatrix} \check{q}_{11} & \check{q}_{12} \\ \check{q}_{21} & \check{q}_{22} \end{pmatrix}, \quad (P_3(x)S_3)^{-1} &= \begin{pmatrix} , \check{q}_{11}^3 & \check{q}_{12}^3 \\ \check{q}_{21}^3 & \check{q}_{22}^3 \end{pmatrix}. \end{aligned}$$

Recall from Section 3 the system within the new coordinate frame has the form:

$$V_m' = B_m V_m + N_m.$$

For illustrative purposes, we list the first few terms of the nonlinearity $\tilde{N}_m$:

$$N_1 = \begin{pmatrix} \check{q}_{12} n_1 \\ \check{q}_{22} n_1 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} \check{q}_{12}^3 n_3 \\ \check{q}_{22}^3 n_3 \end{pmatrix}$$

where

$$\begin{aligned} n_1 &= -r(|aq_{11} + bq_{12}|^2 (aq_{11} + bq_{12}) + |a_3 q_{11}^3 + b_3 q_{12}^3|^2 (aq_{11} + bq_{12}) + \cdots) \\ n_3 &= -r((\mathbf{aq_{11} + bq_{12}})^{\mathbf{3}} + |aq_{11} + bq_{12}|^2 (a_3 q_{11}^3 + b_3 q_{12}^3) + \cdots) \end{aligned}$$

Here was used the fact that

$$\begin{aligned} g_1 &= |u_1|^2 u_1 + |u_3|^2 u_1 + \cdots \\ g_3 &= (u_1)^3 + |u_1|^2 u_3 + \cdots, \end{aligned}$$

where the occurrences of $u_.$ have been replaced by the appropriate $(a_. q_{.,.} + b_. q_{.,.})$. The term above that is boldfaced plays an important role in later computations.

We are now ready to perform the normal form transform. We set $q = V_1 = (a, b)$ which corresponds to the first mode, and $w = (V_3, V_5, ...)$, which corresponds to the remaining modes. Recall from Section 4 that (q, w) satisfy

$$\partial_x q = B_1 q + \tilde{N}_1(x, q, w), \tag{A.5}$$

$$\partial_x w = Lw + G(x, q, w) + H(x, q). \tag{A.6}$$

The terms in the nonlinearity $\tilde{N}_m, m \geq 3$, that only depend on the first mode, q are contained in $H(x, q)$. This corresponds to the bold part of n_3 above. The rest belongs to $G(x, q, w)$. As stated in Section. 4 the goal is to make the term $H(x, q)$ small. This is done iteratively for each $m > 1$. We perform the first iteration of this calculation which starts at $p = 3$. Thus we consider $w_3 = V_3$, and we try to eliminate $H_3(x, q)$ which corresponds the first block diagonal entry of $H(x, q)$. We set

$$\tilde{w}_3 = w_3 + J_3(x, q, q, q),$$

where J_3 satisfies

$$B_3 J_3(x, q) = \frac{d}{dx} J_3(x, q) + H_3(x, q). \tag{A.7}$$

J_3 and H_3 have the form:

$$H_3(x, q) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h_i(x, q) = \sum_{k=0}^{3} h_{i,k}(x) y_k(q) \tag{A.8}$$

$$J_3(x, q) = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, \quad j_i(x, q) = \sum_{k=0}^{3} j_{i,k}(x) y_k(q), \tag{A.9}$$

and in this example

$$\vec{h} = \begin{pmatrix} h_{1,0} \\ h_{1,1} \\ h_{1,2} \\ h_{1,3} \\ h_{2,0} \\ h_{2,1} \\ h_{2,2} \\ h_{2,3} \end{pmatrix} = r \begin{pmatrix} \check{q}_{12}^3 (q_{11})^3 \\ \check{q}_{12}^3 (q_{11})^2 q_{12} \\ \check{q}_{12}^3 (q_{11})(q_{12})^2 \\ \check{q}_{12}^3 (q_{12})^3 \\ \check{q}_{22}^3 (q_{11})^3 \\ \check{q}_{22}^3 (q_{11})^2 q_{12} \\ \check{q}_{22}^3 (q_{11})(q_{12})^2 \\ \check{q}_{22}^3 (q_{12})^3 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} j_{1,0} \\ j_{1,1} \\ j_{1,2} \\ j_{1,3} \\ j_{2,0} \\ j_{2,1} \\ j_{2,2} \\ j_{2,3} \end{pmatrix} \tag{A.10}$$

and

$$y_k(q) = a^{3-k} b^k, \quad k = 0 \ldots 3. \tag{A.11}$$

After substituting (A.8) and (A.9) into (A.7), and making use of the fact that $a'(x) = \lambda_1 a(x) + c_1 b(x) + \text{h.o.t}$ and $b'(x) = -\lambda_1 b(x) + \text{h.o.t.}$ and collecting in terms of a and b gives us a natural choice for eight linear ODEs:

$$\frac{d}{dx}\vec{j} = A\vec{j} + \vec{h} \tag{A.12}$$

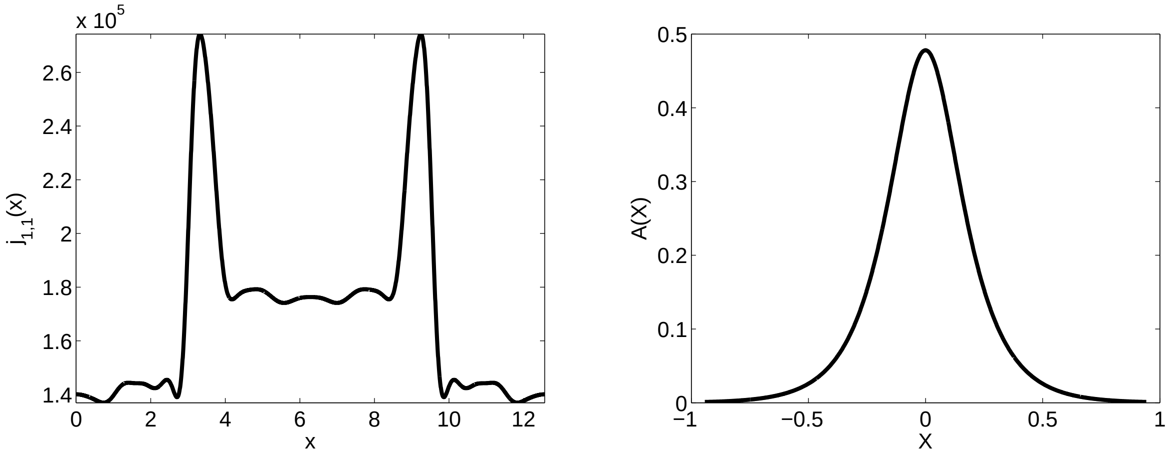

Figure 13: Left: The first element of the matrix $J_3(x,q)$ which satisfies (A.12) and is used to eliminate $H_3(x,q)$ which corresponds to the first block element of the matrix $H(x,q)$ in Equation (A.6). Right: Approximate solution of the NLS-equation.

where

$$A = \begin{pmatrix} \lambda_3 - 3\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3c_1 & \lambda_3 - \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2c_1 & \lambda_3 + \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_1 & \lambda_3 + 3\lambda_1 & 0 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & -\lambda_3 - 3\lambda_1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 & -3c_1 & -\lambda_3 - \lambda_1 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & -2c_1 & -\lambda_3 + \lambda_1 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & -c_1 & -\lambda_3 + 3\lambda_1 \end{pmatrix},$$

where $\lambda_1 = 0.0681$ and $\lambda_3 = 0.106\mathrm{i}$ are eigenvalues corresponding to B_1 and B_3 respectively, $c_1 = 0.001$, and $c_3 = 0.230$ (see Eq. A.4). We solve these equations in Fourier space, see the left panel of Figure 13.

Section 5 describes how to compute an approximate modulating pulse solution, which has the form

$$A(X) = \pm\sqrt{\frac{2s_1}{-s_3}}\,\mathrm{sech}(\sqrt{s_1}X)$$

where $s_1 = \lambda_1^2/\varepsilon^2$ and s_3 is the coefficient of the $|A|^2A$ term in Equation (5.7). In this example, it is

$$s_3 = \frac{1}{2\pi}\int_0^{4\pi} -r(x)\check{q}_{22}(x)|q_{11}(x)|^2 q_{11}(x)dx$$

See the right panel of Figure 13.

B From the physical equations to the studied model

The equation

$$\partial_x^2 E(x,t) = \partial_t^2((1+\alpha(x))E(x,t) + \beta(x)|E(x,t)|^2 E(x,t)) \tag{B.1}$$

with L-periodic α and β for an $L > 0$, is one of the most used models in the description of electromagnetic waves in one-dimensional photonic crystals, cf. [4, Eq. 3.48, 3.49]. From a mathematical point of view (B.1) has the disadvantage of being a quasi-linear system. In this section we will explain that from a modeling point of view (B.1) is equivalent to the semi-linear wave equation

$$\partial_t^2 E(x,t) = \partial_x^2 E(x,t) + \rho(x)E(x,t) + r(x)|E(x,t)|^2 E(x,t) \tag{B.2}$$

which is more tractable by analytic and numerical methods again with L-periodic ρ, r.

In this section we use variables like P, M, etc. for different objects than in the previous sections in order to keep the standard notation since there is no possibility of interchanging the objects.

Maxwell's equations (cf. [8]) in media are given by

$$\begin{aligned} \nabla \cdot (E+P) &= \tilde{\rho}, && \text{(B.3)} \\ \nabla \times E &= \partial_t(B+M), && \text{(B.4)} \\ \nabla \times B &= -\partial_t(E+P) + j, && \text{(B.5)} \\ \nabla \cdot (B+M) &= 0, && \text{(B.6)} \end{aligned}$$

where E is the electric, B the magnetic field, P the polarization, M the magnetization, j the electric current density, and $\tilde{\rho}$ the electric charge density, where by rescaling all coefficients have been set to one.

In photonic crystals there are no free charges, no electric current, and no magnetization, i.e., $\tilde{\rho} = 0$, $j = 0$ and $M = 0$.

By differentiating (B.5) w.r.t t and substituting $\partial_t B$ via (B.4) yields

$$\Delta E - \nabla(\nabla \cdot E) = \partial_t^2 E + \partial_t^2 P \tag{B.7}$$

where we additionally used $\nabla \times \nabla \times u = -\Delta u + \nabla(\nabla \cdot u)$. For polarized light, i.e., in the one-dimensional situation (B.7) simplifies into

$$\partial_x^2 E = \partial_t^2 E + \partial_t^2 P. \tag{B.8}$$

In order to close (B.8) the polarization P has to be expressed in terms of E.

In the simplest model the nuclei of the atoms are fixed and the centers of the electrons move like a nonlinear oscillator. This simple modeling finally leads to a system

$$\begin{aligned} \partial_x^2 E &= \partial_t^2 E + \partial_t^2 P, && \text{(B.9)} \\ P &= \textstyle\sum_{j=1}^N P_j, && \text{(B.10)} \\ \partial_t^2 P_j + \omega_j^2 P_j + r_j P_j |P_j|^2 &= d_j E && \text{(B.11)} \end{aligned}$$

with constants ω_j, r_j, d_j and where N is the number of different kinds of molecules.

The argument to come to (B.1) is as follows. For $E = E_0 e^{i\omega t}$, Equation (B.11) possesses solutions $P_j = P_{0j} e^{i\omega t}$ with

$$-\omega^2 P_{0j} + \omega_j^2 P_{0j} + r_j P_{0j}|P_{0j}|^2 = d_j E_0. \tag{B.12}$$

For photonic crystals the parameters ω_j^2, r_j, and d_j depend periodically on x. For small E_0 (B.12) can be solved w.r.t. P_{0j}, i.e., we have a constitutive law

$$P_{0j}(x,\omega) = \alpha_j(x,\omega)E_{0j}(x,\omega) + \beta_j(x,\omega)E_{0j}^3(x,\omega) + \ldots. \tag{B.13}$$

For ω in the optical window the changes in $\alpha_j(x,\omega)$ are negligible w.r.t. ω, i.e., the relation (B.13) is modeled independently of ω, or equivalently $\omega =: \tilde{\omega}_j$ is fixed. Then multiplying (B.12) by $e^{i\omega t}$ yields the relation

$$(-\tilde{\omega}_j^2 + \omega_j^2 + r_j|P_j|^2)P_j = d_j E,$$

which can be inverted for small E, i.e.,

$$P_j = \frac{d_j}{\omega_j^2 - \tilde{\omega}_j^2} E - r_j \left(\frac{d_j}{\omega_j^2 - \tilde{\omega}_j} \right)^3 |E|^2 E + \ldots. \tag{B.14}$$

Inserting this into (B.8) yields (B.1). On the other hand, we can proceed as follows. We replace first $\partial_t^2 P$ in (B.8) via (B.10) and (B.11), i.e.,

$$\partial_x^2 E = \partial_t^2 E + \sum_{j=1}^{N} (d_j E - \omega_j^2 P_j - r_j P_j |P_j|^2). \tag{B.15}$$

Using (B.14) to replace P_j in (B.15) yields (B.2) when terms of order $\mathcal{O}(|P_j|^5)$ are neglected.

C Proof of Lemma 3.5

We consider

$$\partial_x^2 u(x) - \rho(x)u(x) = -m^2\omega^2 u(x).$$

By Floquet's theorem the solutions are given by $u(x) = e^{\lambda x} p(x)$ with $p(x) = p(x + 2\pi)$ and λ such that $|\mathrm{Im}\lambda| \leq 1/2$. We use Fourier series

$$p(x) = \sum_{k\in\mathbb{Z}} p_k e^{ikx} \qquad \text{and} \qquad \rho(x) = \sum_{k\in\mathbb{Z}} \rho_k e^{ikx}$$

and obtain

$$((\lambda + ik)^2 + m^2\omega^2)p_k = \sum_{l\in\mathbb{Z}} \rho_{k-l} p_l.$$

We have to find λ in such a way that this infinite-dimensional system of equations possesses non-trivial solutions $(p_k)_{k\in\mathbb{Z}}$.

$\rho = 0$**:** For $\rho = 0$ we have a two-dimensional kernel. There is a unique k_0 such that

$$\lambda_+ = \mathrm{i}m\omega - \mathrm{i}k_0 \qquad \text{and} \qquad \lambda_- = -\mathrm{i}m\omega + \mathrm{i}k_0$$

with $-\frac{1}{2} < \operatorname{Im}\lambda_\pm \leq \frac{1}{2}$ and $-\frac{1}{2} \leq \operatorname{Im}\lambda_- < \frac{1}{2}$. By the choice $|\operatorname{Im}\lambda_\pm| \leq \frac{1}{2}$ we have that m and k_0 are proportional.

$\rho \neq 0$**:** In order to handle the situation $\rho \neq 0$ we use a Lyapunov-Schmidt like reduction. We introduce the deviation μ of λ_+ by

$$\lambda_+ = \mathrm{i}m\omega + \mathrm{i}k_0 + \mu$$

We obtain for $k = \pm k_0$ that

$$\text{(C.1)} \qquad (2\mathrm{i}m\omega\mu + \mu^2 + \rho_0)p_{k_0} + \rho_{2k_0}p_{-k_0} = \sum_{l\in\mathbb{Z}/\{k_0,-k_0\}} \rho_{k_0-l}p_l.$$

$$\text{(C.2)} \qquad (-2\mathrm{i}m\omega\mu + \mu^2 + \rho_0)p_{-k_0} + \rho_{-2k_0}p_{k_0} = \sum_{l\in\mathbb{Z}/\{k_0,-k_0\}} \rho_{-k_0-l}p_l$$

and for $k \neq \pm k_0$ that

$$(2\mathrm{i}m\omega\mu - 2m(k-k_0) - 2\mathrm{i}\mu(k-k_0) + \mu^2 + (k-k_0)^2)p_k = \sum_{l\in\mathbb{Z}} \rho_{k-l}p_l.$$

For m sufficiently large the last set of equations can be solved w.r.t. p_k for $k \neq \pm k_0$ as a function of μ, p_{k_0} and p_{-k_0} with norm of order $\mathcal{O}(1/m)$. More precisely, since $\rho \in H^1_{\mathrm{per}}$ implies $(\rho_k)_{k\in\mathbb{Z}} \in \ell^1$ and since ℓ^1 is closed under convolution, the implicit function theorem can be applied in the space ℓ^1 and

$$\|(p_k)_{k\in\mathbb{Z}\setminus\{-k_0,k_0\}}\|_{\ell^1} \leq \mathcal{O}(1/m)\mathcal{O}(|\mu| + |p_{k_0}| + |p_{-k_0}|).$$

This resolution can be substituted in the bifurcation equations (C.1) and (C.2) which now only include μ, p_{k_0} and p_{-k_0}. Since (C.1) and (C.2) depend linearly on p_{k_0} and p_{-k_0} they can be written as $M\begin{pmatrix} p_{k_0} \\ p_{-k_0}\end{pmatrix} = 0$. We find nontrivial solutions if we choose μ in such a way that the determinant of the matrix M vanishes. We find

$$|\mu| \leq \mathcal{O}(1/m)\mathcal{O}(|\rho_{2k_0}| + |\rho_{-2k_0}|).$$

Hence we found for the time-periodic part in case of no Jordan-block that

$$p(x) = (e^{ik_0x} + h.o.t.)c_1 + (e^{-ik_0x} + h.o.t.)c_2$$

where $h.o.t.$ means $\mathcal{O}(1/m)$ in $\ell^1(s)$. This leads to the transformation

$$U(x) = \begin{pmatrix} e^{\mathrm{i}k_0x} + h.o.t. \\ \mathrm{i}k_0(e^{\mathrm{i}k_0x} + h.o.t.)\end{pmatrix} v_1(x) + \begin{pmatrix} e^{-\mathrm{i}k_0x} + h.o.t. \\ -\mathrm{i}k_0(e^{-\mathrm{i}k_0x} + h.o.t.)\end{pmatrix} v_2(x)$$

and finally to

$$Q_m(x) = \begin{pmatrix} e^{\mathrm{i}k_0x} + h.o.t. & e^{-\mathrm{i}k_0x} + h.o.t. \\ \mathrm{i}k_0(e^{\mathrm{i}k_0x} + h.o.t.) & -\mathrm{i}k_0(e^{-\mathrm{i}k_0x} + h.o.t.)\end{pmatrix}.$$

This immediately gives the assertion since $k_0 \sim m$. □

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