

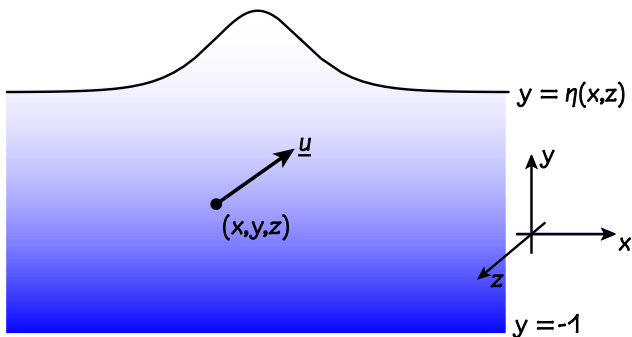
FULLY LOCALISED THREE-DIMENSIONAL SOLITARY WATER WAVES ON BELTRAMI FLOWS WITH STRONG SURFACE TENSION

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Beltrami water waves



- Governing equations in a frame moving with the wave:

$$\operatorname{div} \underline{u} = 0, \quad \operatorname{curl} \underline{u} = a \underline{u}$$

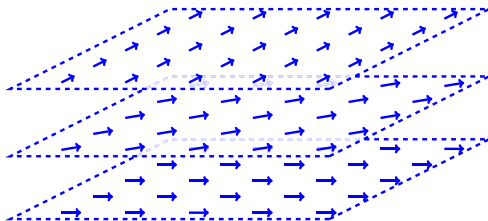
$$\underline{u} \cdot \underline{j} |_{y=-1} = 0, \quad \underline{u} \cdot \underline{n} |_{y=\eta} = 0$$

$$\frac{1}{2} |\underline{u}|^2 \Big|_{y=\eta} + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_z = \frac{1}{2} |c|^2$$

- \underline{u} and $p = -\frac{1}{2} |\nabla \underline{u}|^2 - y$ satisfy the steady Euler equation

Trivial solution

$$\eta^* = 0, \quad \underline{u}^* = c_1 \begin{pmatrix} \cos ay \\ 0 \\ \sin ay \end{pmatrix} + c_3 \begin{pmatrix} -\sin ay \\ 0 \\ \cos ay \end{pmatrix}$$



- Write $\underline{u} = \underline{u}^* + \underline{v}$, so that

$$\operatorname{div} \underline{v} = 0, \quad \operatorname{curl} \underline{v} = a \underline{v}$$

$$\underline{v} \cdot \underline{j}|_{y=-1} = 0, \quad \underline{v} \cdot \underline{n}|_{y=\eta} + \underline{u}^* \cdot \underline{n}|_{y=\eta} = 0$$

$$\frac{1}{2} |\underline{v}|^2 \Big|_{y=\eta} + \underline{v} \cdot \underline{u}^* \Big|_{y=\eta} + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_z = 0$$

Modelling

- Solutions of the form

$$\eta = A e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} = (x, z)^T, \quad \mathbf{k} = (k_1, k_3)^T$$

satisfy the dispersion relation

$$g(\mathbf{k}) = 0$$

- $g(\underline{0}) = 0$ if

$$c_0^2 = \frac{1}{2}a \tan \frac{1}{2}a, \quad c_1 = c_0 \cos \frac{1}{2}a, \quad c_3 = c_0 \sin \frac{1}{2}a$$

and g is otherwise positive if β is sufficiently large

- The Ansatz

$$c_1 = (c_0 - \varepsilon^2) \cos \frac{1}{2}a, \quad c_3 = (c_0 - \varepsilon^2) \sin \frac{1}{2}a,$$

$$\eta(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z) + O(\varepsilon^4)$$

leads to the KP-I equation

$$\zeta_{xx} - \zeta - \frac{D_3^2}{D_1^2} \zeta - 3\zeta^2 = 0$$

KP-I equation

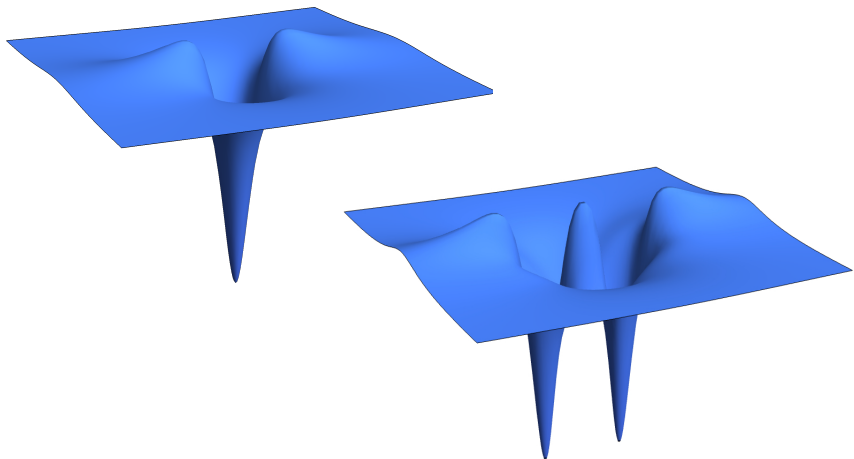
Liu, Wei & Yang have shown that (\star) has an infinite family $\{\zeta_k\}$ of symmetric lump solutions of the form

$$\zeta_k(x, z) = -2\partial_x^2 \log \tau_k(x, z)$$

- τ_k is a polynomial of degree $k(k+1)$ given by an explicit formula
- They prove that ζ_1 and ζ_2 are nondegenerate, and claim that ζ_k is always nondegenerate:

$$\left. \begin{aligned} \partial_x^2(\partial_x^2 \zeta - \zeta + 6\zeta_k \zeta) - \partial_y^2 \zeta &= 0 \\ \zeta(-x, -z) &= \zeta(x, z) \\ \lim_{|(x,z)| \rightarrow \infty} \zeta(x, z) &= 0 \end{aligned} \right\} \Rightarrow \zeta = 0$$

KP lumps



Fully localised solitary waves

For sufficiently large β , sufficiently small ε and

$$c_0^2 = \frac{1}{2}a \tan \frac{1}{2}a,$$

$$c_1 = (c_0 - \varepsilon^2) \cos \frac{1}{2}a, \quad c_3 = (c_0 - \varepsilon^2) \sin \frac{1}{2}a,$$

the Beltrami water-wave problem has a family $\{\eta_k\}$ of fully localised solitary wave solutions which satisfy

$$\eta_k(-x, -z) = \eta_k(x, z)$$

and

$$\eta_k(x, z) = \varepsilon^2 \zeta_k(\varepsilon x, \varepsilon^2 z) + o(\varepsilon^2)$$

uniformly over $(x, z) \in \mathbb{R}^2$

ZCS formulation for water waves

- Let ϕ be the unique solution of the boundary-value problem

$$\left. \begin{aligned} \Delta\phi &= 0, \\ \phi_y|_{y=-1} &= 0, \\ \phi|_{y=\eta} &= \xi \end{aligned} \right\} \Rightarrow \begin{aligned} \operatorname{div} \underline{v} &= 0, \\ \operatorname{curl} \underline{v} &= \underline{0}, \\ \underline{v} \cdot \underline{j}|_{y=-1} &= 0 \end{aligned}$$

and set $\underline{v} = \operatorname{grad} \phi$

- Zakharov (1968), Craig & Sulem (1993): We can formulate the problem in terms of η and ξ
- Dirichlet-Neumann operator: $G(\eta)\xi = \operatorname{grad} \phi|_{y=\eta} \cdot \underline{N}$
- Free-surface boundary conditions:

$$G(\eta)\xi + \underline{c} \cdot \nabla\eta = 0,$$

$$\frac{1}{2}|\nabla\xi|^2 - \frac{(G(\eta)\xi + \nabla\eta \cdot \nabla\xi)^2}{2(1 + |\nabla\eta|^2)}$$

$$- \underline{c} \cdot \nabla\xi + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_z = 0$$

Generalised ZCS formulation

- For $\underline{F} = (F_1, F_2, F_3)^\top$ define

$$\underline{F}_{\parallel} = (F_1, F_3)^\top + F_2 \nabla \eta|_{y=\eta}$$

- For $\underline{G} = (G_1, G_3)^\top$ write

$$\underline{G} = \nabla \Phi + \nabla^\perp \Psi,$$

where

$$\Phi = \Delta^{-1}(\nabla \cdot \underline{G}), \quad \Psi = \Delta^{-1}(\nabla^\perp \cdot \underline{G})$$

- G. & Horn (2020): We can formulate the problem in terms of η and $\xi = \Delta^{-1}(\nabla \cdot \underline{v}_{\parallel})$
- Motivation: Benjamin (1984), Castro & Lannes (2015)

Generalised ZCS formulation

- Let \underline{A} be the unique solution of the boundary-value problem

$$\left. \begin{aligned} \operatorname{curl} \operatorname{curl} \underline{A} &= a \operatorname{curl} \underline{A}, \\ \operatorname{div} \underline{A} &= 0, \\ \underline{A} \wedge \underline{j}|_{y=-1} &= \underline{0}, \\ \underline{A} \cdot \underline{n}|_{y=\eta} &= 0, \\ (\operatorname{curl} \underline{A})_{\parallel} &= \nabla \xi - a \nabla^{\perp} \Delta^{-1} (\nabla \cdot \underline{A}_{\parallel}^{\perp}) \end{aligned} \right\} \Rightarrow \begin{aligned} \operatorname{div} \underline{v} &= 0, \\ \operatorname{curl} \underline{v} &= a \underline{v}, \\ \underline{v} \cdot \underline{j}|_{y=-1} &= 0 \end{aligned}$$

and set $\underline{v} = \operatorname{curl} \underline{A}$

- Generalised Dirichlet-Neumann operator: $H(\eta)\xi = \operatorname{curl} \underline{A}|_{y=\eta} \cdot \underline{N}$
- Free-surface boundary conditions:

$$H(\eta)\xi + \underline{u}^* \cdot \underline{N}|_{y=\eta} = 0,$$

$$\frac{1}{2} |K(\eta)\xi|^2 - \frac{(H(\eta)\xi + K(\eta)\xi \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)}$$

$$+ K(\eta)\xi \cdot \underline{u}_{\parallel}^*|_{y=\eta} + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \right)_z = 0,$$

where $K(\eta)\xi = \nabla \xi - a \nabla^{\perp} \Delta^{-1} (H(\eta)\xi)$

Generalised Dirichlet-Neumann operator

$$\begin{aligned}
 H(\eta)\xi &= \operatorname{curl} \underline{A}|_{y=\eta} \cdot \underline{N}, & \operatorname{curl} \operatorname{curl} \underline{A} &= a \operatorname{curl} \underline{A}, \\
 & & \operatorname{div} \underline{A} &= 0, \\
 & & \underline{A} \wedge \underline{j}|_{y=-1} &= \underline{0}, \\
 & & \underline{A} \cdot \underline{n}|_{y=\eta} &= 0, \\
 & & (\operatorname{curl} \underline{A})_{\parallel} &= \nabla \xi - a \nabla^{\perp} \Delta^{-1} (\nabla \cdot \underline{A}_{\parallel}^{\perp})
 \end{aligned}$$

• For $s \geq 2$ the mappings

$$H(\cdot) : H^{s+\frac{1}{2}}(\mathbb{R}^2) \rightarrow \mathcal{L}(\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^2), \check{H}^{s-\frac{3}{2}}(\mathbb{R}^2))$$

and

$$H(\cdot)^{-1} : H^{s+\frac{1}{2}}(\mathbb{R}^2) \rightarrow \mathcal{L}(\check{H}^{s-\frac{3}{2}}(\mathbb{R}^2), \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^2))$$

are analytic at the origin, where

$$\dot{H}^t(\mathbb{R}^2) = \{f \in L_{loc}^2(\mathbb{R}^2) : \nabla f \in (H^{t-1}(\mathbb{R}^2))^2\},$$

$$\check{H}^t(\mathbb{R}^2) = \{f \in L_{loc}^2(\mathbb{R}^2) : \Delta^{-1} f \in \dot{H}^{t+2}(\mathbb{R}^2)\}$$

Irrotational limit

$$H(\eta)\xi = \text{curl } \underline{A}|_{y=\eta} \cdot \underline{N},$$

$$\text{curl curl } \underline{A} = \underline{0},$$

$$\text{div } \underline{A} = 0,$$

$$\underline{A} \wedge \underline{j}|_{y=-1} = \underline{0},$$

$$\underline{A} \cdot \underline{n}|_{y=\eta} = 0,$$

$$(\text{curl } \underline{A})_{\parallel} = \nabla \xi$$

- We find that $\text{curl } \underline{A} = \text{grad } \varphi$, where

$$\Delta \varphi = 0,$$

$$\varphi_y|_{y=-1} = 0,$$

$$\varphi|_{y=\eta} = \xi$$

(because $(\text{grad } \varphi)_{\parallel} = \nabla(\varphi|_{y=\eta})$)

- Hence

$$\begin{aligned} H(\eta)\xi &= \text{grad } \varphi|_{y=\eta} \cdot \underline{N} \\ &= G(\eta)\xi \end{aligned}$$

Formulation as a single equation

- Eliminating ξ from

$$H(\eta)\xi + \underline{u}^* \cdot \underline{N}|_{y=\eta} = 0,$$

$$\frac{1}{2}|K(\eta)\xi|^2 - \frac{(H(\eta)\xi + K(\eta)\xi \cdot \nabla\eta)^2}{2(1 + |\nabla\eta|^2)}$$

$$+ K(\eta)\xi \cdot \underline{u}_h^*|_{y=\eta} + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_z = 0$$

yields

$$J(\eta) := \frac{1}{2}|N(\eta, c)|^2 - \frac{(-\underline{u}^*|_{y=\eta} \cdot \underline{N} + \nabla\eta \cdot N(\eta, c))^2}{2(1 + |\nabla\eta|^2)} + N(\eta, c) \cdot \underline{u}_h^*|_{y=\eta} \\ + \eta - \beta \left(\frac{\eta_x}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_x - \beta \left(\frac{\eta_z}{(1 + |\nabla\eta|^2)^{\frac{1}{2}}} \right)_z = 0, \quad (\star)$$

where

$$N(\eta, c) = -\nabla(H(\eta))^{-1}(\underline{u}^*|_{y=\eta} \cdot \underline{N}) + a\nabla^\perp \Delta^{-1}(\underline{u}^*|_{y=\eta} \cdot \underline{N})$$

- Study (\star) for $\eta \in H^3(\mathbb{R}^2)$

Dispersion relation

Solutions of the form

$$\eta = A e^{i \underline{k} \cdot \underline{x}}, \quad \underline{x} = (x, z)^T, \quad \underline{k} = (k_1, k_3)^T$$

satisfy

$$g(\underline{k}) := -\frac{1}{|\underline{k}|^2} (a(\underline{c}_0 \cdot \underline{k}^\perp)(\underline{c}_0 \cdot \underline{k}) + f(|\underline{k}|^2)(\underline{c}_0 \cdot \underline{k})^2) + 1 + \beta |\underline{k}|^2 = 0,$$

where

$$f(|\underline{k}|^2) = \begin{cases} (|\underline{k}|^2 - a^2)^{1/2} \coth(|\underline{k}|^2 - a^2)^{1/2}, & |\underline{k}|^2 \geq a^2, \\ (a^2 - |\underline{k}|^2)^{1/2} \cot(a^2 - |\underline{k}|^2)^{1/2}, & |\underline{k}|^2 \leq a^2 \end{cases}$$

- g is an analytic function of $(k_1, \frac{k_3}{k_1})$ at the origin
- $g(0,0) = 0$ if

$$c_0^2 = \frac{1}{2} a \tan \frac{1}{2} a, \quad c_1 = c_0 \cos \frac{1}{2} a, \quad c_3 = c_0 \sin \frac{1}{2} a;$$

$(0,0)$ is a strict local minimum if $\beta > \beta_0 := \frac{1}{2a^2} (-\cos a + a \operatorname{cosec} a)$

- $g > 0$ otherwise if $\beta > \beta^* := \frac{1}{a} \left(-\frac{1}{a} \cot a + \operatorname{cosec}^2 a \right) \tan \frac{1}{2} a$

Reduction

$$\eta(x, z) \sim \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z)$$

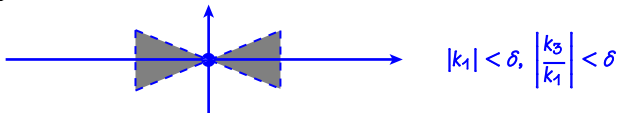
- Write

$$c_1 = (c_0 - \varepsilon^2) \cos \frac{1}{2}a, \quad c_3 = (c_0 - \varepsilon^2) \sin \frac{1}{2}a,$$

- Write

$$\eta_1 = \chi\left(D_1, \frac{D_3}{D_1}\right)\eta, \quad \eta_2 = \left(1 - \chi\left(D_1, \frac{D_3}{D_1}\right)\right)\eta,$$

where χ is the characteristic function of this set:



- Consider

$$\chi\left(D_1, \frac{D_3}{D_1}\right)J(\eta_1 + \eta_2) = 0, \quad (1)$$

$$\left(1 - \chi\left(D_1, \frac{D_3}{D_1}\right)\right)J(\eta_1 + \eta_2) = 0 \quad (2)$$

- Solve (2) for $\eta_2 = \eta_2(\eta_1)$ and insert into (1) to yield the reduced equation

$$\chi\left(D_1, \frac{D_3}{D_1}\right)J(\eta_1 + \eta_2(\eta_1)) = 0$$

Reduction

- Write

$$\eta_1(x, z) = \varepsilon^2 \zeta(\varepsilon x, \varepsilon^2 z)$$

- The reduced equation is

$$\varepsilon^{-2} g(\varepsilon D_1, \varepsilon \frac{D_3}{D_1}) \zeta + 2\zeta + \chi(\varepsilon D_1, \varepsilon \frac{D_3}{D_1}) \left(c_d \zeta^2 + \underline{O}(\varepsilon^{\frac{1}{2}} \|\zeta\|_Y) \right) = 0 \quad (\star)$$

for $\zeta \in B_R(0) \subseteq \chi(\varepsilon D_1, \varepsilon \frac{D_3}{D_1}) Y$, where

$$Y = \left\{ u \in L^2(\mathbb{R}^2) : \|u\|_Y^2 = \int_{\mathbb{R}^2} \left(1 + k_1^2 + \frac{k_3^2}{k_1^2} \right) |D(k)|^2 dk < \infty \right\}$$

- Replace ζ with $\chi(\varepsilon D_1, \varepsilon \frac{D_3}{D_1}) \zeta$ and work in the fixed space Y
- In the limit $\varepsilon \rightarrow 0$ we obtain the KP-I equation

$$\zeta_{xx} - \zeta - \frac{D_3^2}{D_1^2} \zeta - 3\zeta^2 = 0$$

from (\star)

Existence

- The symmetric solutions ζ_k of

$$\zeta_{xx} - \zeta - \frac{D_3^2}{D_1^2} \zeta - 3\zeta^2 = 0$$

are non-degenerate in the class of solutions symmetric under $\zeta(x, z) \mapsto \zeta(-x, z)$, $\zeta(x, z) \mapsto \zeta(x, -z)$

- The reduced equation (\star) has this symmetry (inherited from the original problem)

- Apply an implicit-function theorem argument to (\star) in the space

$$Y_\varepsilon = \{\zeta \in Y : \zeta(x, z) = \zeta(-x, -z)\}$$

- (\star) has symmetric solitary waves for small values of ε

