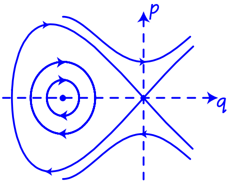
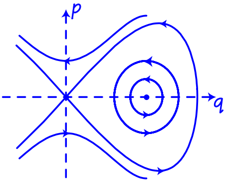
THE (HAMILTONIAN) KIRCHGÄSSNER REDUCTION

Mark Groves

Homoclinic orbits in Hamiltonian systems

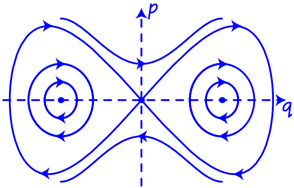




$$q_x = \partial_p H = q,$$
$$p_x = -\partial_q H = \mu q - C q^2$$

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}\mu q^2 + C q^3$$

$$q(x) = \frac{3\mu}{2C}\,\mathrm{sech}^2\,\frac{\sqrt{\mu}x}{2}$$



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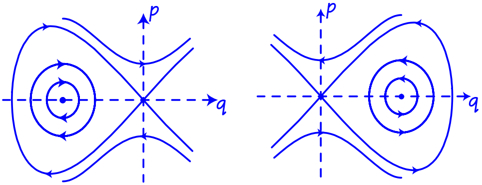
Homoclinic orbits in Hamiltonian systems



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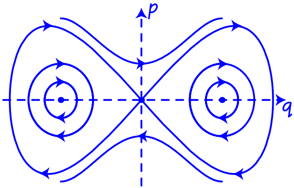
$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}\mu q^2 + C q^3$$
$$+ O(\mu|(q,p)|^2 + |(q,p)|^3)$$

$$q(x) = \frac{3\mu}{2C}\,\mathrm{sech}^2 \frac{\sqrt{\mu}x}{2}x + O(\mu^{3/2})$$



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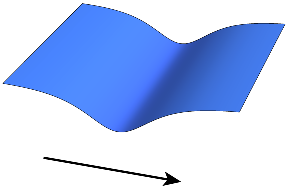
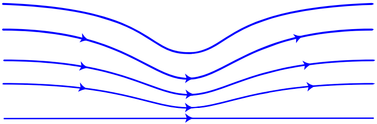
$$q(x) = \sqrt{\frac{2\mu}{C}}\,\mathrm{sech}\,\sqrt{\mu}x + O(\mu)$$

- Persistence by reversibility

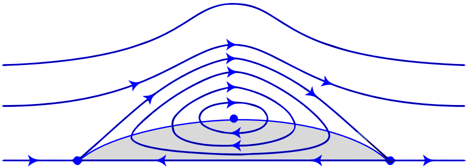
Solitary water waves

- Wave of depression with strong surface tension (Kirchgässner 1988)



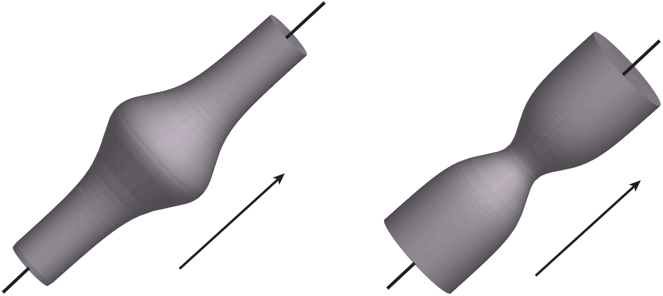


- Gravity wave of elevation ‘riding’ a flow with large constant vorticity (Kozlov, Kuznetsov & Lokharu 2020)



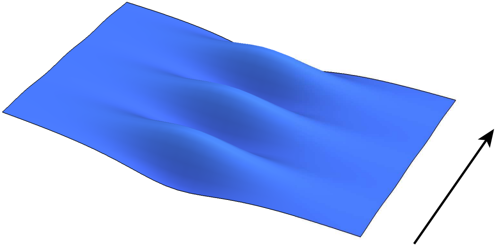
Solitary waves on a ferrofluid jet

- Axisymmetric surface waves on a ferrofluid jet surrounding a current-carrying wire (Groves & Nilsson 2018)



Three-dimensional travelling water waves

- Periodic gravity-capillary travelling water waves with a localised transverse profile (Groves 2001)



Framework

Begin with a spatial dynamics problem

$$u_x = f^\mu(u) \qquad \text{in } \Gamma,$$
$$B^\mu(u) = 0 \qquad \text{on } \partial\Gamma$$

- The domain Γ is fixed
- Reversible: there is a reverser S with

$$f^\mu(Su) = -f^\mu(u), \qquad B^\mu(Su) = \pm B^\mu(u)$$

- Hamiltonian: there is a Hamiltonian system (X, Ω^μ, H^μ) with

$$\Omega^\mu|_u(f^\mu(u), v) + \Upsilon(B^\mu(u), v) = dH^\mu[u](v)$$

for all v

Step 1: Linearise the boundary conditions

$$u_x = f^\mu(u) \qquad \text{in } \Gamma,$$
$$B^\mu(u) = 0 \qquad \text{on } \partial\Gamma$$

Use a near-identity change of variables

$$u = v + G^\mu(v), \qquad G^\mu(v) = O(|\mu|\|v\| + \|v\|^2)$$

to linearise and de-parametrise the boundary conditions:

$$v_x = L(v) + N^\mu(v) \qquad \text{in } \Gamma,$$
$$L_B(v) = 0 \qquad \text{on } \partial\Gamma,$$

where $L = df^0[0]$ and $L_B = dB^0[0]$

- Hamiltonian: $(X, \Omega^\mu_{lbc}, H^\mu_{lbc})$ with $H^\mu_{lbc}(v) = H^\mu(v + G^\mu(v))$ and $\Omega^0_{lbc}|_0 = \Omega^0|_0$

Step 2: Centre-manifold reduction

$$v_x = L(v) + N^{\mu}(v) \qquad \text{in } X,$$

where

$$L : \mathcal{D} \subset X \to X, \qquad N \in C^{\infty}(\mathcal{D} \times \mathbb{R}, X)$$

- L is a Riesz spectral operator (its generalised eigenvectors form a Riesz basis $\{e_i\}$ for X):
  - $X = \{x_i e_i : \{x_i\} \in \ell^2\}$
  - $D = \{x_i e_i : \{\lambda_i x_i\} \in \ell^2\}$ (if all eigenvalues are simple)
- Write $X = X_c \oplus X_h$, where

$$X_c = sp\{e_i : \lambda_i \in i\mathbb{R}\}, \qquad X_h = \overline{sp\{e_i : \lambda_i \notin i\mathbb{R}\}}$$

  and $\dim X_c < \infty$
- Spectral projections:

$$P_S v = \sum_{i : \lambda_i \in S} \langle v, f_i \rangle e_i,$$

  where $\{f_i\}$ is the dual Riesz basis

Step 2: Centre-manifold reduction

$$v_x = L(v) + N^\mu(v) \qquad \text{in } X,$$

where

$$L : \mathcal{D} \subset X \longrightarrow X, \qquad N \in C^\infty(\mathcal{D} \times \mathbb{R}, X)$$

- All small globally bounded solutions lie on

$$M_c = \{(v_c, v_h) : v_h = h^\mu(v_c)\},$$

where $h^\mu(v_c) = O(|\mu| \|v_c\| + \|v_c\|^2)$

- Reduced equation:

$$v_{cx} = Lv_c + N^\mu(v_c + h^\mu(v_c))$$

  - Hamiltonian: (X_c, Q_{cm}, H_{cm}) with $H^\mu_{cm}(v_c) = H^\mu_{lbc}(v_c + h^\mu(v_c))$ and $Q^O_{cm}|_O = Q^O_{lbc}|_O$

Step 3: Coordinates for the centre manifold

- Consider the special case $v_c = qe + pf$, where $Le = 0$, $Lf = e$
- Use a symplectic basis for X_c, such that $\Omega^0|_0(e, f) = 1$:

$$\Omega^0_{cm}|_0(v_c^1, v_c^2) = \Omega_{can}(v_c^1, v_c^2) = q^1 p^2 - p^1 q^2$$

- Use a near-identity 'Darboux' change of variables

$$v_c = w_c + D^\mu(w_c), \qquad D^\mu(w_c) = O(|\mu|\|w_c\| + \|w_c\|^2)$$

  to transform $(X_c, \Omega_{cm}, H_{cm})$ into $(X_c, \Omega_{can}, H_{red})$, where $H^\mu_{red}(w_c) = H^\mu_{cm}(w_c + D^\mu(w_c))$

- Possibly use a further near-identity '(partial) normal-form' transformation that does not change Ω_{can}
- Final result:

$$q_x = \partial_p H^\mu_{red}(q, p),$$
$$p_x = -\partial_q H^\mu_{red}(q, p)$$

  - Reversible: $S(q, p) = (q, -p)$ if $Se = e$, $Sf = -f$

How to calculate H_{red}?

- $$\begin{aligned} u &= v + G^\mu(v) \\ &= v_c + h^\mu(v_c) + G^\mu(v_c + h^\mu(v_c)) \\ &= w_c + D^\mu(w_c) + h^\mu(w_c + D^\mu(w_c)) \\ &\quad + G^\mu(w_c + D^\mu(w_c) + h^\mu(w_c + D^\mu(w_c))) \\ &=: w_c + k^\mu(w_c), \qquad k^\mu(w_c) = O(|\mu| \|w_c\| + \|w_c\|^2) \end{aligned}$$
- $H^\mu_{red}(w_c) = H^\mu(w_c + k^\mu(w_c))$
- Coefficients in the Maclaurin series

  $$k^\mu(w_c) = \sum_{\substack{i+j_1+j_2 \geq 2 \\ j_1+j_2 \geq 1}} k_{i,j_1,j_2} \mu^i q^{j_1} p^{j_2}$$

  can be calculated by substituting $u = w_c + k^\mu(w_c)$ into

  $$\begin{aligned} u_x &= f^\mu(u) && \text{in } \Omega, \\ B^\mu(u) &= 0 && \text{on } \partial\Omega \end{aligned}$$
- Simplify using

  $$\Omega^\mu|_u(f^\mu(u), v) + \Upsilon(B^\mu(u), v) = dH^\mu[u](v)$$

How to calculate H_{red}?

- The coefficient of q^3 in H^μ_{red} is

$$C_3 = H^0_3(e,e,e)$$

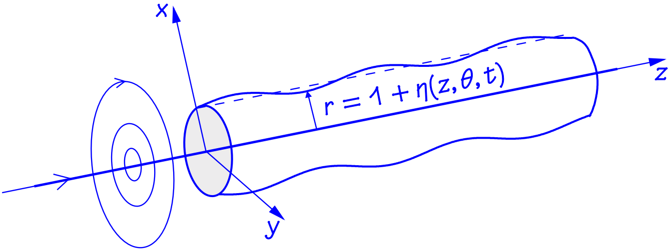
- If $C_3 = 0$ the coefficient of q^4 in H^μ_{red} is

$$C_4 = H^0_4(e,e,e,e) + \frac{3}{2}H^0_3(e,e,k_{0,20}),$$

where

$$f^0_1 k_{0,20} = -f^0_2(e,e),$$
$$B^0_1 k_{0,20} = -B^0_2(e,e)$$

Solitary waves on a ferrofluid jet



- Axisymmetric travelling waves:
  - $\eta(z,\theta,t) = \eta(z-ct)$
  - The hydrodynamic problem decouples
- Formulate in terms of (η, ϕ)

Solitary waves on a ferrofluid jet

The governing equations

$$
\begin{aligned}
&\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, && 0 < r < 1 + \eta(z),\\
&\phi_r = 0, && r = 0,\\
&\eta_z + \phi_r - \phi_z\eta_z = 0, && r = 1 + \eta(z),\\
&-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - a\frac{T'(\eta)}{1+\eta} && \\
&\qquad + \beta\left(\frac{1}{(1+\eta)(1+\eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - 1\right) = 0, && r = 1 + \eta(z)
\end{aligned}
$$

follow from the formal variational principle

$$
\delta \int \left\{ \int_0^{1+\eta} \left(\frac{1}{2} r\phi_r^2 + \frac{1}{2} r\phi_z^2 - r\phi_z \right) dr \right.
\left. - aT(\eta) + \beta(1+\eta)(1+\eta_z^2)^{1/2} - \frac{1}{2}\beta(1+\eta)^2 \right\} dz = 0
$$

Solitary waves on a ferrofluid jet

- 'Flatten' the equations with the transformation

$$s = \frac{r}{1+\eta(z)}, \qquad \Phi(s,z) = \phi(r,z)$$

- The flattened equations follow from the variational principle

$$\delta \int L(\eta, \Phi; \eta_z, \Phi_z)\, dz = 0,$$

where

$$L(\eta, \Phi; \eta_z, \Phi_z) =$$
$$\int_0^1 \left\{ \frac{1}{2}\left(s\Phi_s^2 + \left[\Phi_z - \frac{s\eta_z\Phi_s}{1+\eta}\right]^2 (1+\eta)^2 s \right) - \left(\Phi_z - \frac{s\eta_z\Phi_s}{1+\eta}\right)(1+\eta)^2 s \right\} ds$$
$$- aT(\eta) + \beta(1+\eta)(1+\eta_z^2)^{1/2} - \frac{1}{2}\beta(1+\eta)^2$$

Solitary waves on a ferrofluid jet

- Variational principle: $\delta \int L(\eta, \Phi; \eta_z, \Phi_z)\, dz = 0$
- Legendre transform:

$$\omega = \frac{\delta L}{\delta \eta_z}, \xi = \frac{\delta L}{\delta \Phi_z} \quad \Rightarrow \quad \eta_z = \eta_z(\eta, \omega, \Phi, \xi), \Phi_z = \Phi_z(\eta, \omega, \Phi, \xi)$$

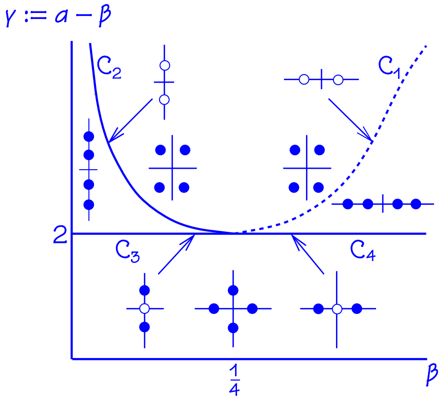
and

$$H(\eta, \omega, \Phi, \xi) = \eta_z \omega + \int_0^1 s \Phi_z \xi \, ds - L(\eta, \Phi, \eta_z, \Phi_z)$$
$$= \int_0^1 \left\{ \frac{1}{2} \left[\frac{\xi}{(1+\eta)^2} + 1 \right]^2 (1+\eta)^2 s - \frac{1}{2} s \Phi_s^2 \right\} ds$$
$$+ aT(\eta) - (1+\eta)\sqrt{\beta^2 - W^2} + \frac{1}{2}\beta(1+\eta)^2,$$

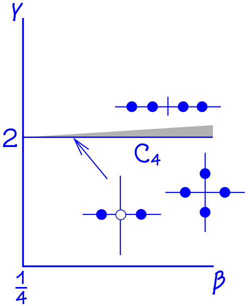
where $W = \frac{1}{1+\eta}\left(\omega + \frac{1}{1+\eta}\int_0^1 s^2 \Phi_s \xi \, ds\right)$ satisfies $|W| < \beta$

- $\mathcal{Q}\big((\eta_1, \Phi_1, \omega_1, \xi_1), (\eta_2, \Phi_2, \omega_2, \xi_2)\big) = \omega_2 \eta_1 - \omega_1 \eta_2 + \int_0^1 (\xi_2 \Phi_1 - \xi_1 \Phi_2) r \, dr$
- Reversibility: $S(\eta, \omega, \Phi, \xi) = (\eta, -\omega, -\Phi, \xi)$

Eigenvalues



Reduction



$\beta = \beta_0 > \frac{1}{4},$

$\gamma = 2 + \mu^2$

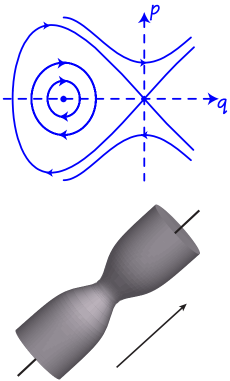
Solitary waves on a ferrofluid jet

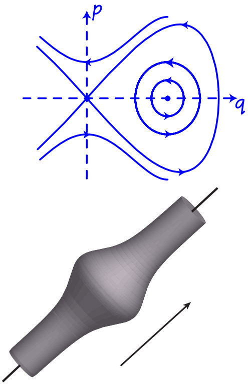
- Write

$$\beta > \tfrac{1}{4}, \quad \gamma = 2 + \mu$$

- After scaling z and q one obtains the reduced Hamiltonian

$$H^{\mu}_{\text{red}} = \tfrac{1}{2}p^2 - \tfrac{1}{2}\mu q^2 + \tfrac{1}{6}(aT'''(0) - 6)q^3 + O(\mu|(q,p)|^2 + |(q,p)|^4)$$





Riesz bases

- A sequence $\{e_i\}$ is a Riesz basis for X if and only if

$$\overline{sp\{e_i\}} = X, \qquad \sum_{i=1}^{n} |a_i|^2 \lesssim \left\| \sum_{i=1}^{n} a_i e_i \right\|^2 \lesssim \sum_{i=1}^{n} |a_i|^2$$

  for all $a_1, \ldots, a_n$ (uniformly in n).

- Suppose that $\{e_i\}$ is a Riesz basis for X and that $\{f_i\}$ is a sequence with the properties that

  (i) $\sum_{i=1}^{\infty} a_i f_i = 0 \quad \Rightarrow \quad a_i = 0, \, i = 1, 2, \ldots$

  (ii) $\sum_{i=1}^{\infty} \|e_i - f_i\|^2 < \infty.$

  Then $\{f_i\}$ is a Riesz basis for X.

- The spectrum of a linear operator with compact resolvent consists of isolated eigenvalues of finite multiplicity and its generalised eigenvectors have the property (i).

Example (from water waves)

Let $\beta_0 > \frac{1}{3}$. The operator $L : D \subset X \longrightarrow X$, where

$$X = \left\{(\eta, \omega, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0,1) \times \bar{L}^2(0,1)\right\},$$

$$D = \Big\{(\eta, \omega, \Phi, \Psi) \in \mathbb{R} \times \mathbb{R} \times \bar{H}^2(0,1) \times \bar{H}^1(0,1) :$$

$$\Phi_z(0) = 0,\ -\Phi_z(1) + \frac{1}{\beta_0}\left(\omega - \int_0^1 z\Phi_z\,dz\right) = 0\Big\}$$

and

$$L\begin{pmatrix}\eta \\ \omega \\ \Phi \\ \Psi\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta_0}\left(\omega - \int_0^1 z\Phi_z\,dz\right) \\ 0 \\ \Psi \\ -\Phi_{zz} - \frac{1}{\beta_0}\left(\omega - \int_0^1 z\Phi_z\,dz\right)\end{pmatrix},$$

has compact resolvent. We will show that it is a Riesz spectral operator.

Spectrum

- $Le = 0$, $Lf = e$, where

$$e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 \\ \beta_0 - \frac{1}{3} \\ -\frac{1}{2}z^2 + \frac{1}{6} \\ 0 \end{pmatrix}$$

- $Le_n = \lambda_n e_n$, where $\{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is a real sequence with
  - $\lambda_{-n} = -\lambda_n$,
  - $\lambda_n \in ((n-1)\pi, n\pi)$ for $n = 1, 2, \ldots,$
  - $\lambda_n^2 = n^2\pi^2 - \dfrac{2}{\beta_0} + o\left(\dfrac{1}{n}\right)$ for large n.

  The eigenvectors are

$$e_n = \begin{pmatrix} \frac{1}{\lambda_n}\sin\lambda_n \\ 0 \\ \frac{1}{\lambda_n}\cos(\lambda_n z) - \frac{1}{\lambda_n^2}\sin\lambda_n \\ \cos(\lambda_n z) - \frac{1}{\lambda_n}\sin\lambda_n \end{pmatrix}.$$

Auxiliary result

The set

$$\left\{ \begin{pmatrix} (n\pi)^{-1} \cos(n\pi z) \\ \cos(n\pi z) \end{pmatrix} \right\}_{n \in \mathbb{Z}\setminus\{0\}}$$

is a Riesz basis for $\bar{H}^1(0,1) \times \bar{L}^2(0,1)$.

- $\left\{ \sqrt{2}\cos(n\pi z) \right\}_{n=1}^{\infty}$ and $\left\{ \sqrt{2}(n\pi)^{-1}\cos(n\pi z) \right\}_{n=1}^{\infty}$ are orthonormal bases for $\bar{L}^2(0,1)$ and $\bar{H}^1(0,1)$ respectively.

- $$sp \left\{ \begin{pmatrix} (n\pi)^{-1} \cos(n\pi z) \\ \cos(n\pi z) \end{pmatrix} \right\}_{n \in \mathbb{Z}\setminus\{0\}} = sp \left\{ \begin{pmatrix} \sqrt{2}(n\pi)^{-1} \cos(n\pi z) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}\cos(n\pi z) \end{pmatrix} \right\}_{n=1}^{\infty}$$

- $$\sum_{|i|=1}^{n} |a_i|^2 \lesssim \left\| \sum_{|i|=1}^{n} a_i \begin{pmatrix} (i\pi)^{-1} \cos(i\pi z) \\ \cos(i\pi z) \end{pmatrix} \right\|^2 \lesssim \sum_{|i|=1}^{n} |a_i|^2$$

for all $a_{\pm 1}, \ldots, a_{\pm n}$ (uniformly in n).

Corollary

The set

$$\{e, f\} \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ (n\pi)^{-1} \cos(n\pi z) \\ \cos(n\pi z) \end{pmatrix} \right\}_{n \in \mathbb{Z} \setminus \{0\}}$$

is a Riesz basis for $X = \mathbb{R} \times \mathbb{R} \times \bar{H}^1(0,1) \times \bar{L}^2(0,1)$

Riesz basis

The set $\{e, f\} \cup \{e_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ is a Riesz basis for $\mathcal{X}$.

- $$\alpha e + \beta f + \sum_{n \in \mathbb{Z}\setminus\{0\}} \gamma_n e_n = 0 \quad \Rightarrow \quad \alpha = \beta = \gamma_n = 0.$$

- Furthermore

$$\|e - e\|^2 + \|f - f\|^2 + \sum_{n \in \mathbb{Z}\setminus\{0\}} \underbrace{\left\| e_n - \begin{pmatrix} 0 \\ 0 \\ \sqrt{2}(n\pi)^{-1} \cos(n\pi z) \\ \sqrt{2} \cos(n\pi z) \end{pmatrix} \right\|^2}_{\lesssim \left|\lambda_{|n|} - |n|\pi\right| = O(\frac{1}{n})} < \infty$$