Diffusive stability of planar wave trains in reaction-diffusion systems against nonlocalized perturbations

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Motivation

Reaction-diffusion system on the plane

\[ u_t = D \Delta u + f(u), \quad u(x, y, t) \in \mathbb{R}^n, \]

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad f : \mathbb{R}^n \to \mathbb{R}^n \text{ and } D \text{ positive diagonal matrix}. \]

- Paradigmatic class for **pattern formation**:
  - stripes
  - spots
  - rings
  - spirals
  - ...

- **Applications** in chemistry, biology, ecology, ...

- **Stability**: resistance of patterns against small perturbations?
Motivation

Stability theory well-established in one spatial dimension

\[ u_t = Du_{xx} + f(u), \quad u(x, t) \in \mathbb{R}^n, \quad x \in \mathbb{R}. \]

Questions

- Planar patterns `inherit' stability from 1D-counterpart?
- More spatial dimensions \( \Rightarrow \) larger class of (nonlocal) perturbations?

Answer questions for planar wave trains (stripes)
Step towards stability of spirals (\( \approx \) planar wave trains away from core)
Set-up

- General (nonlinear) stability theory in reaction-diffusion systems
- Nonlinear stability of wave trains on the line
- Nonlinear stability of planar wave trains
Nonlinear stability in reaction-diffusion systems
Nonlinear stability

Reaction-diffusion system on $\mathbb{R}^m$

\[ u_t = D\Delta u + f(u), \quad u(x, t) \in \mathbb{R}^n, \quad \Delta = \partial_{x_1}^2 + \ldots + \partial_{x_m}^2 \]

- **Stationary** solution: $u_*(x) = u_*(x_1, \ldots, x_m)$
- Stability **Ansatz**: $u(x, t) = u_*(x) + v(x, t)$

**Perturbation equation**

\[ \nu_t = u_t = D\Delta(u_* + \nu) + f(u_* + \nu) \]
\[ = D\Delta \nu + f'(u_*)\nu + f(u_* + \nu) - f(u_*) - f'(u_*)\nu \]

Generally $\mathcal{N}(\nu) \sim \nu^2$.

$u_*$ is (asymptotically) **nonlinearly stable** if

\[ \nu(0) \text{ small} \implies \lim_{t \to \infty} \nu(t) = 0. \]
Nonlinear stability by spectral approximation

If $\mathcal{L}$ and $\mathcal{N}$ well-behaved, proceed as in ODE-setting:

**Integral formulation**

\[ v_t = \mathcal{L}v + \mathcal{N}(v) \iff v(t) = e^{\mathcal{L}t}v_0 + \int_0^t e^{\mathcal{L}(t-s)}\mathcal{N}(v(s))\,ds \]

**Strong spectral stability $\implies$ nonlinear stability**

\[
\sup \Re(\sigma(\mathcal{L})) < 0 \implies \|e^{\mathcal{L}t}\| \leq Ce^{-\delta t} \\
\|v_0\| \text{ small} \implies \|v(t)\| \leq \tilde{C}e^{-\delta t}
\]

\[
\sup \Re(\sigma(\mathcal{L})) > 0 \text{ yields (nonlinear) instability.}
\]
Weak spectral stability

If $u_* \neq \text{constant}$, expect $0 \in \sigma(L)$:

$$
0 = D\Delta u_* + f(u_*) \implies 0 = D\Delta \partial_{x_i} u_* + f'(u_*) \partial_{x_i} u_* = L[\partial_{x_i} u_*].
$$

Weak spectral stability!

- Continuous spectrum up to $i\mathbb{R} \Rightarrow \text{algebraic} \text{ decay } \|e^{Lt}\| \leq Ct^{-\delta}$.
- $0 \in \sigma(L)$ eigenvalue $\Rightarrow \|e^{Lt}\| \leq C$.

Often cannot close nonlinear iteration argument $\implies \text{instability?}$
Weak spectral stability

Well-known example

\[ u_t = \Delta u + u^p, \quad u(x, t) \in \mathbb{R}, \quad \Delta = \partial_{x_1}^2 + \ldots + \partial_{x_m}^2. \]

Linearize about \( u_* = 0 \):

\[ \mathcal{L} = \Delta, \quad \sigma(\mathcal{L}) = (-\infty, 0]. \]

- **Nonlinear stability** of \( u_* \) for \( p > 1 + \frac{2}{m} \) (in \( L^1 \cap L^\infty \)-space).
- **Instability** of \( u_* \) for \( p < 1 + \frac{2}{m} \).

Intuition

- More spatial dimensions improve decay of semigroup.
- Higher nonlinearities improve stability.
Weak spectral stability

\[ u_t = D \Delta u + f(u), \quad u(x, t) \in \mathbb{R}^n, \]

Translational invariance

- Every translate \( u_*(x + \psi) \) is a solution.
- Recall: if \( u_* \neq \text{constant} \), expect \( 0 \in \sigma(L) \).

Approach [Doelman, Sandstede, Scheel, Schneider '09]

Modulation Ansatz: \( u(x + \psi(x, t), t) = u_*(x) + \tilde{v}(x, t) \) yields

\[ \tilde{v}_t = \mathcal{L}\tilde{v} + \widetilde{N}(\tilde{v}, \psi) \]

- \( \psi(x, t) \) accounts for translation.
- If \( 0 \in \sigma(L) \) eigenvalue, residual \( \tilde{v}(t) \) decays exponentially.
- If continuous spectrum up to \( i\mathbb{R} \), gain algebraic factor \( t^{-\beta} \).
Nonlinear stability of wave trains
Wave trains

**Traveling wave train** $u_1(x, t) = u_*(kx - \omega t)$ solves

$$u_t = Du_{xx} + f(u), \quad u(x, t) \in \mathbb{R}^n,$$

with $u_*$ 1-periodic and $k, \omega \in \mathbb{R}$.

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**Planar wave train** $u_2(x, y, t) = u_*(kx - \omega t)$ solves

$$u_t = D\Delta u + f(u), \quad \Delta = \partial_x^2 + \partial_y^2.$$
Nonlinear stability of wave trains on the line
Wave trains on the line

Co-moving frame $\zeta = kx - \omega t$

**Stationary solution** $u_*(\zeta)$ to

$$u_t = Dk^2 u_{\zeta\zeta} + \omega u_{\zeta} + f(u).$$

**Linearization**

$$\mathcal{L}_1 v = Dk^2 v_{\zeta\zeta} + \omega v_{\zeta} + f'(u_*(\zeta))v.$$

- **$\mathcal{L}_1$ periodic** $\implies \sigma(\mathcal{L}_1)$ parameterized over $S^1$.
- **Translational invariance** $\implies 0 \in \sigma(\mathcal{L}_1)$.

![Graphs showing weak spectral stability and instability](image)
Wave trains on the line

Weak spectral stability $\implies$ nonlinear stability

**Initial data:**

$$u(\zeta, 0) = u_*(\zeta) + v_0(\zeta).$$

**Theorem 1** [Johnson, Zumbrun ‘11]

$$\|v_0\|_{L^1 \cap H^3 \text{ small}} \implies \|u(t) - u_*\|_{H^3} \leq \frac{C}{\sqrt{1 + t}}.$$

**Theorem 2** [Jung ‘12]

$$\|v_0\|_{H^2}, \sup_{\zeta \in \mathbb{R}} \|v_0(\zeta)e^{\zeta^2/2M}\| \text{ small} \implies \|u(\zeta, t) - u_*(\zeta)\| \leq Ce^{-\frac{|\zeta + \alpha t|^2}{M(1+t)}} \frac{1}{\sqrt{1 + t}}.$$

- Proofs using **modulation Ansatz**.
- Pointwise estimates yield **diffusive Gaussian-like decay**.
- $\alpha$ group velocity.
Wave trains on the line

With (nonlocal) phase modulation

Initial data:
\[ u(\zeta, 0) = u_*(\zeta + \psi_0(\zeta)) + v_0(\zeta), \quad \lim_{\zeta \to \pm \infty} \psi_0(\zeta) = \psi_{\pm}. \]

Theorem [Sandstede, Scheel, Schneider, Uecker ‘12]
\[ \|v_0\|_{H^2(2)}, \|\partial_\zeta \psi_0\|_{H^2(2)}, |\psi_+ - \psi_-| \text{ small} \implies \exists \text{ explicit } \psi(\zeta, t) : \|u(\cdot - \psi(\cdot, t), t) - u_*\|_{L^\infty} \leq \frac{C}{(1 + t)^{3/4}}. \]

Extensions:
- [Jung, Zumbrun ‘16]: pointwise estimates
- [Iyer, Sandstede ‘16]: phase offset \( \psi_+ - \psi_- \) large
Nonlinear stability of planar wave trains
Planar wave trains

\[ u_2(x, y, t) = u_*(kx - \omega t) \] solves

\[ u_t = D\Delta u + f(u), \quad \Delta = \partial^2_x + \partial^2_y. \]

Co-moving frame \( \zeta = kx - \omega t \): stationary solution \( u_* \) to

\[ u_t = D \left( k^2 u_{\zeta\zeta} + u_{yy} \right) + \omega u_{\zeta} + f(u). \]

Linearization

\[ \mathcal{L}_2 v = D \left( k^2 v_{\zeta\zeta} + v_{yy} \right) + \omega v_{\zeta} + f'(u_*(\zeta))v. \]

- \( \mathcal{L}_2 \) is \( \zeta \)-periodic and \( y \)-homogeneous \( \implies \sigma(\mathcal{L}_2) \) parameterized over \( S^1 \times \mathbb{R} \).
- Translational invariance \( \implies 0 \in \sigma(\mathcal{L}_2) \).
Planar wave trains

**Weak spectral stability:**

0 simple eigenvalue of \( \mathcal{L}_1 = Dk^2 \partial_{\zeta \zeta} + \omega \partial_{\zeta} + f'(u_*) \) in \( L^2_{\text{per}}([0, 1]) \);

**Inherit weak spectral stability from 1D-wave train?** Yes (about 0) iff

\[
\langle u_{\text{ad}}, Du_*' \rangle_{L^2_{\text{per}}} \langle u_{\text{ad}}, u_*' \rangle_{L^2_{\text{per}}} > 0,
\]

where \( u_{\text{ad}} \) spans \( \ker(\mathcal{L}_1^*) \) in \( L^2_{\text{per}}([0, 1]) \).
Planar wave trains

Weak spectral stability $\implies$ nonlinear stability

Initial data:
$$u(\zeta, y, 0) = u_*(\zeta) + v_0(\zeta, y).$$

Localized perturbations [dR, Sandstede ‘17]

$$\sup_{\zeta, y \in \mathbb{R}} \left( \| v_0(\zeta, y) \| + \| \partial_\zeta v_0(\zeta, y) \| \right) e^{\frac{\zeta^2+y^2}{M}} \text{ small}$$

$$\implies \| u(\zeta, y, t) - u_*(\zeta) \| \leq C e^{\frac{-|\zeta+\alpha t|^2+y^2}{M(1+t)}} \frac{1}{1+t}.$$

- Diffusive decay;
- $\alpha$ group velocity;
- Faster decay wrt 1D.
Planar wave trains

Weak spectral stability $\implies$ nonlinear stability

Initial data:

$$u(\zeta, y, 0) = u_*(\zeta) + v_0(\zeta, y).$$

Nonlocalized perturbations [dR, Sandstede '17]

$$\sup_{\zeta, y \in \mathbb{R}} \left( \|v_0(\zeta, y)\| + \|\partial_\zeta v_0(\zeta, y)\| \right) e^{\frac{|\beta \zeta + \gamma y|^2}{M}} \text{ small}$$

$$\implies \|u(\zeta, y, t) - u_*(\zeta)\| \leq C e^{-\frac{|\beta \zeta + \alpha \beta t + \gamma y|^2}{M(1+t)}} \frac{1}{\sqrt{1 + t}},$$

with $(\beta, \gamma) \in \mathbb{R}^2$ unit vector.
Planar wave trains

Remarks

- **Nonlocalization** in any spatial direction in $\mathbb{R}^2$.
- Nonlocalization does **not originate** from phase modulation.
- **Trade-off** between localization and decay.

Intuition

- Diffusion smothes out in all spatial directions.
- **Extra** spatial dimension **improves decay**.
**Planar wave trains**

### Proof

- **Pointwise bounds** to exploit spatial structure.  
  → developed by Zumbrun et al (see [Zumbrun, Howard ‘98]).

- **Modulation Ansatz** to account for **translational invariance**.

### Challenge:

- **modulation Ansatz requires** $L^2$-**damping estimate**.  
  → **prohibits** nonlocalization.

### Resolution

- **Alternate** between modulated and nonmodulated coordinates.

- **Control derivatives** through nonmodulated Ansatz.  
  → circumvent damping estimate.

- **Close nonlinear iteration** using modulated Ansatz.
Summary

Nonlinear stability analysis of wave trains from 1D to 2D:

- **Sign criterion:** 2D-stability induced by 1D-stability?

- Extra spatial dimension yields **larger class** of **nonlocal** perturbations.
  \[\rightarrow\] Nonlocality is structurally different (no phase modulation).

Future directions:

- **Different** (nonlocal) perturbations (algebraic weights)
- **Higher** spatial dimensions
- **Spirals**
- Stability against nonlocal perturbations in **1D**?
Thank you for the attention!

- B. de Rijk, B. Sandstede. Diffusive stability against nonlocalized perturbations of planar wave trains in reaction-diffusion systems, *preprint online*