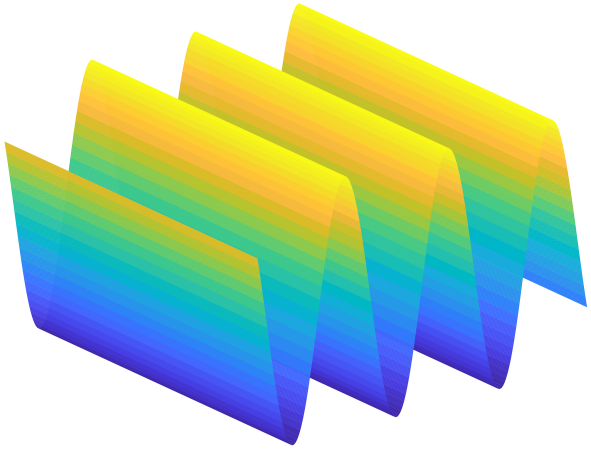
Diffusive stability of planar wave trains in reaction-diffusion systems against nonlocalized perturbations



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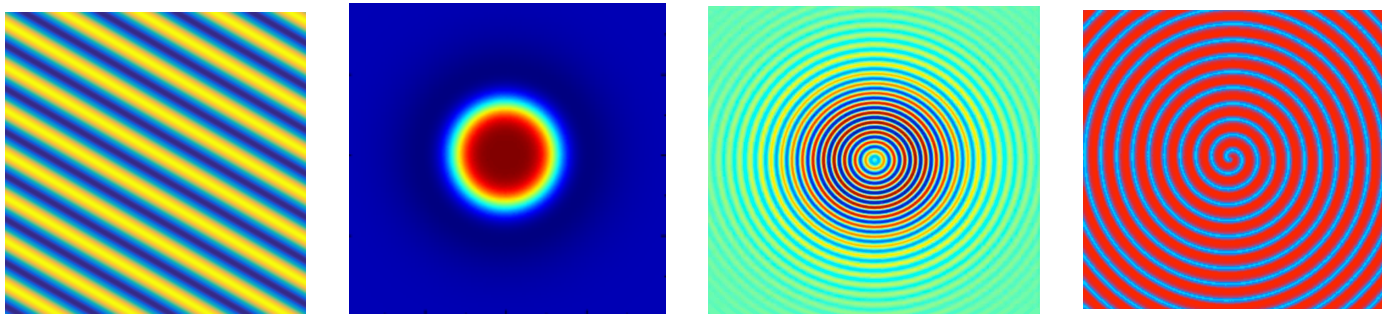
Motivation

Reaction-diffusion system on the plane

$$u_t = D\Delta u + f(u), \qquad u(x, y, t) \in \mathbb{R}^n,$$

$\Delta = \partial_x^2 + \partial_y^2$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and D positive diagonal matrix.

- Paradigmatic class for **pattern formation:**



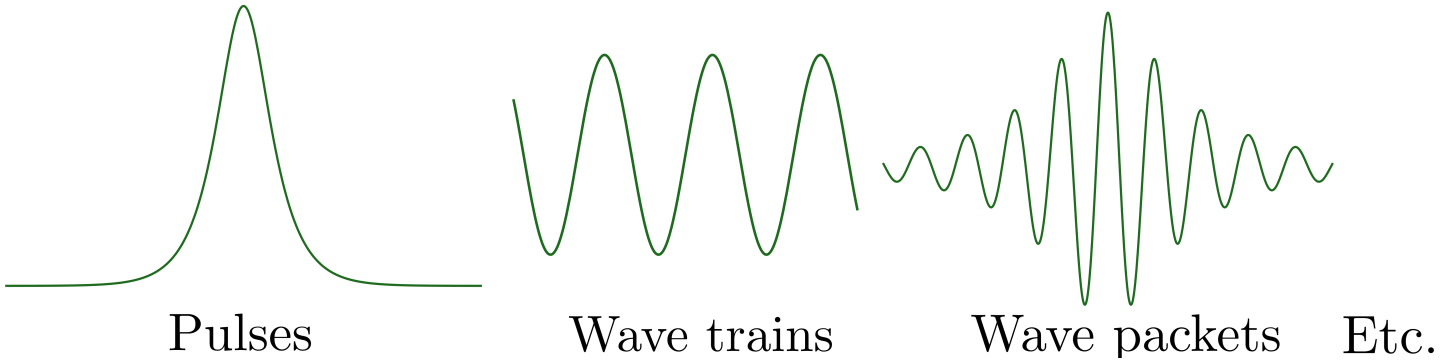
stripes spots rings spirals ...

- **Applications** in chemistry, biology, ecology, ...
- **Stability**: resistance of patterns against small perturbations?

Motivation

Stability theory well-established in **one** spatial dimension

$$u_t = Du_{xx} + f(u), \qquad u(x,t) \in \mathbb{R}^n, \qquad x \in \mathbb{R}.$$



Questions

- Planar patterns **'inherit'** stability from 1D-counterpart?
- More spatial dimensions $\Longrightarrow$ **larger** class of (nonlocal) perturbations?

- Answer questions for **planar wave trains** (stripes)
- Step towards stability of **spirals** ($\approx$ planar wave trains away from core)

Set-up

- **General (nonlinear) stability theory** in reaction-diffusion systems
- Nonlinear stability of **wave trains on the line**
- Nonlinear stability of **planar wave trains**

Nonlinear stability in reaction-diffusion systems

Nonlinear stability

Reaction-diffusion system on $\mathbb{R}^m$

$$u_t = D\Delta u + f(u), \qquad u(x,t) \in \mathbb{R}^n, \qquad \Delta = \partial_{x_1}^2 + \ldots + \partial_{x_m}^2$$

- **Stationary** solution: $u_*(x) = u_*(x_1, \ldots, x_m)$
- Stability **Ansatz**: $u(x,t) = u_*(x) + v(x,t)$

Perturbation equation

$$\begin{aligned} v_t = u_t &= D\Delta(u_* + v) + f(u_* + v) \\ &= \underbrace{D\Delta v + f'(u_*)v}_{\mathcal{L}v} + \underbrace{f(u_* + v) - f(u_*) - f'(u_*)v}_{\mathcal{N}(v)} \end{aligned}$$

Generally $\mathcal{N}(v) \sim v^2$.

u_* is (asymptotically) **nonlinearly stable** if

$$v(0) \text{ small} \Longrightarrow \lim_{t\to\infty} v(t) = 0.$$

Nonlinear stability by spectral approximation

If $\mathcal{L}$ and $\mathcal{N}$ well-behaved, proceed as in ODE-setting:

Integral formulation

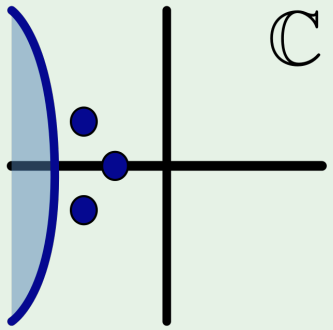
$$v_t = \mathcal{L}v + \mathcal{N}(v) \quad \Longleftrightarrow \quad v(t) = e^{\mathcal{L}t}v_0 + \int_0^t e^{\mathcal{L}(t-s)}\mathcal{N}(v(s))ds$$

Strong spectral stability $\Longrightarrow$ nonlinear stability

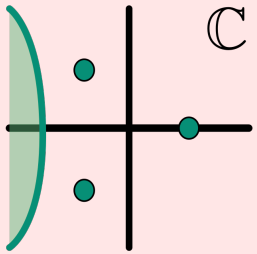
$\sup \mathrm{Re}\,(\sigma(\mathcal{L})) < 0$

$$\Longrightarrow \quad \|e^{\mathcal{L}t}\| \leq Ce^{-\delta t}$$

$$\Longrightarrow \quad \left[\|v_0\| \text{ small } \Rightarrow \|v(t)\| \leq \widetilde{C}e^{-\delta t}\right]$$



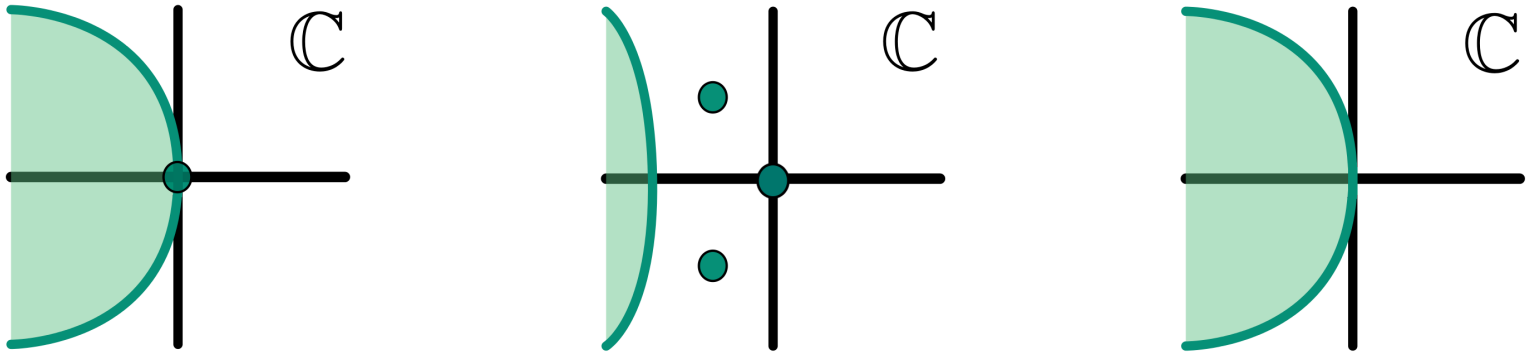
$\sup \mathrm{Re}\,(\sigma(\mathcal{L})) > 0$ yields (nonlinear) instability.



Weak spectral stability

If $u_* \neq$ constant, expect $0 \in \sigma(\mathcal{L})$:

$$0 = D\Delta u_* + f(u_*) \quad \Longrightarrow \quad 0 = D\Delta \partial_{x_i} u_* + f'(u_*)\partial_{x_i} u_* = \mathcal{L}[\partial_{x_i} u_*].$$



Weak spectral stability!

- Continuous spectrum up to $i\mathbb{R} \Rightarrow$ **algebraic** decay $\|e^{\mathcal{L}t}\| \leq Ct^{-\delta}$.
- $0 \in \sigma(\mathcal{L})$ eigenvalue $\Rightarrow \|e^{\mathcal{L}t}\| \leq C$.

Often cannot close nonlinear iteration argument $\Longrightarrow$ **instability?**

Weak spectral stability

Well-known example

$$u_t = \Delta u + u^p, \qquad u(x,t) \in \mathbb{R}, \qquad \Delta = \partial_{x_1}^2 + \ldots + \partial_{x_m}^2.$$

Linearize about $u_* = 0$:

$$\mathcal{L} = \Delta, \qquad \sigma(\mathcal{L}) = (-\infty, 0].$$

- **Nonlinear stability** of u_* for $p > 1 + \frac{2}{m}$ (in $L^1 \cap L^\infty$-space).
- **Instability** of u_* for $p < 1 + \frac{2}{m}$.

Intuition

- **More spatial dimensions** improve decay of semigroup.
- **Higher nonlinearities** improve stability.

Weak spectral stability

$$u_t = D\Delta u + f(u), \qquad u(x,t) \in \mathbb{R}^n,$$

Translational invariance

- Every **translate** $u_*(x+\psi)$ is a solution.
- Recall: if $u_* \neq$ constant, expect $0 \in \sigma(\mathcal{L})$.

Approach [Doelman, Sandstede, Scheel, Schneider '09]

Modulation Ansatz: $u(x+\psi(x,t),t) = u_*(x) + \tilde{v}(x,t)$ yields

$$\tilde{v}_t = \mathcal{L}\tilde{v} + \widetilde{\mathcal{N}}(\tilde{v},\psi)$$

- $\psi(x,t)$ accounts for translation.
- If $0 \in \sigma(\mathcal{L})$ **eigenvalue**, residual $\tilde{v}(t)$ decays **exponentially**.
- If **continuous spectrum** up to $i\mathbb{R}$, gain **algebraic** factor $t^{-\beta}$.

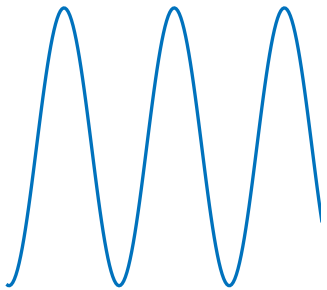
Nonlinear stability of wave trains

Wave trains

Traveling **wave train** $u_1(x,t) = u_*(kx - \omega t)$ solves

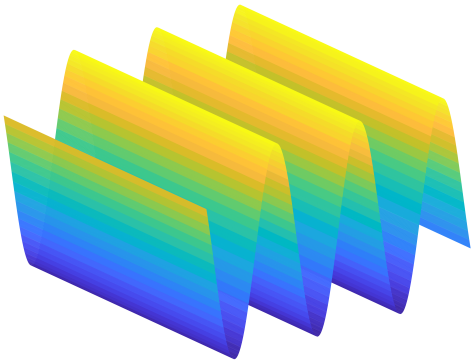
$$u_t = Du_{xx} + f(u), \qquad u(x,t) \in \mathbb{R}^n,$$

with u_* 1-**periodic** and $k, \omega \in \mathbb{R}$.



Planar wave train $u_2(x,y,t) = u_*(kx - \omega t)$ solves

$$u_t = D\Delta u + f(u), \qquad \Delta = \partial_x^2 + \partial_y^2.$$



Nonlinear stability of wave trains on the line

Wave trains on the line

Co-moving frame $\zeta = kx - \omega t$

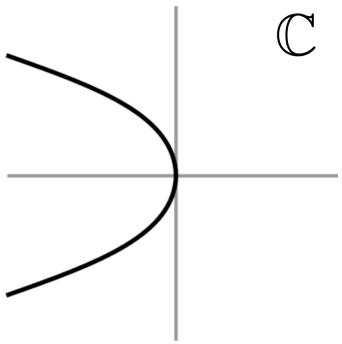
Stationary solution $u_*(\zeta)$ to

$$u_t = Dk^2 u_{\zeta\zeta} + \omega u_\zeta + f(u).$$

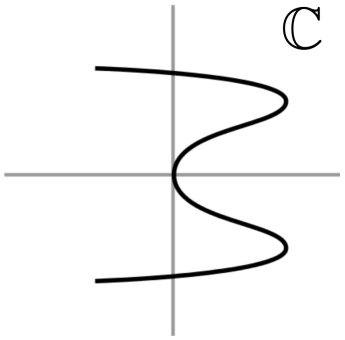
Linearization

$$\mathcal{L}_1 v = Dk^2 v_{\zeta\zeta} + \omega v_\zeta + f'(u_*(\zeta))v.$$

- $\mathcal{L}_1$ **periodic** $\Longrightarrow \sigma(\mathcal{L}_1)$ parameterized over S^1.
- **Translational invariance** $\Longrightarrow 0 \in \sigma(\mathcal{L}_1)$.



weak spectral stability



instability

Wave trains on the line

Weak spectral stability $\Longrightarrow$ nonlinear stability

Initial data:

$$u(\zeta, 0) = u_*(\zeta) + v_0(\zeta).$$

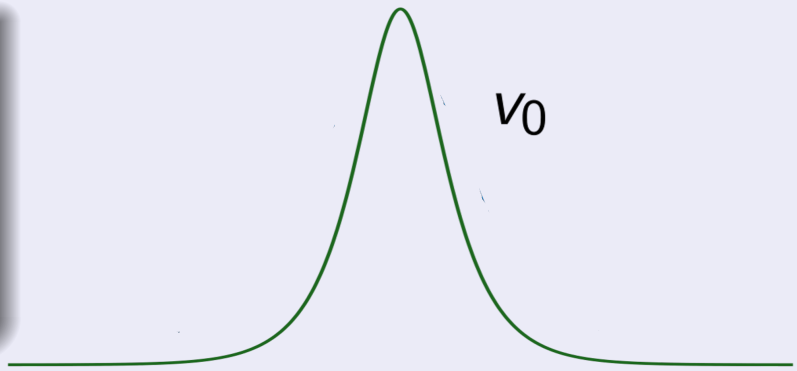
Theorem 1 [Johnson, Zumbrun '11]

$$\|v_0\|_{L^1 \cap H^3} \text{ small} \Longrightarrow \|u(t) - u_*\|_{H^3} \leq \frac{C}{\sqrt[4]{1+t}}.$$

Theorem 2 [Jung '12]

$$\|v_0\|_{H^2}, \sup_{\zeta \in \mathbb{R}} \|v_0(\zeta) e^{\frac{\zeta^2}{M}}\| \text{ small} \Longrightarrow \|u(\zeta, t) - u_*(\zeta)\| \leq C \frac{e^{-\frac{|\zeta + \alpha t|^2}{M(1+t)}}}{\sqrt{1+t}}.$$

- Proofs using **modulation Ansatz**.
- Pointwise estimates yield **diffusive Gaussian-like decay**.
- α **group velocity**.



Wave trains on the line

With (nonlocal) phase modulation

Initial data:

$$u(\zeta,0) = u_*(\zeta + \psi_0(\zeta)) + v_0(\zeta), \qquad \lim_{\zeta\to\pm\infty} \psi_0(\zeta) = \psi_\pm.$$

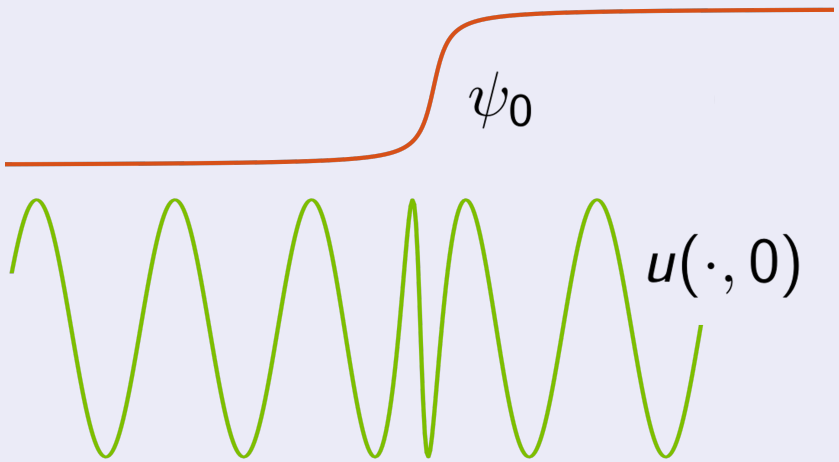
Theorem [Sandstede, Scheel, Schneider, Uecker '12]

$$\|v_0\|_{H^2(2)}, \|\partial_\zeta \psi_0\|_{H^2(2)}, |\psi_+ - \psi_-| \text{ small} \Longrightarrow$$

$$\exists \textbf{ explicit } \psi(\zeta,t) : \|u(\cdot - \psi(\cdot,t),t) - u_*\|_{L^\infty} \leq \frac{C}{(1+t)^{\frac{3}{4}}}.$$

Extensions:

- [Jung, Zumbrun '16]:
  pointwise estimates
- [Iyer, Sandstede '16]:
  phase offset $\psi_+ - \psi_-$ **large**

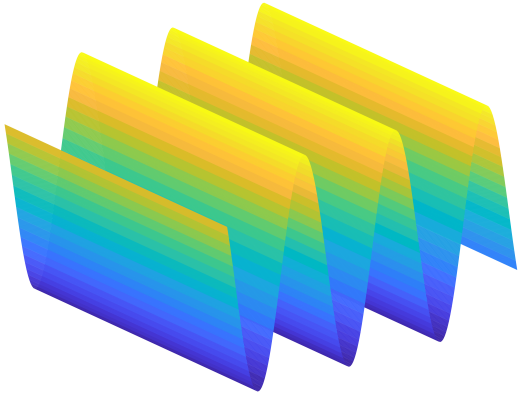


Nonlinear stability of planar wave trains

Planar wave trains

$u_2(x, y, t) = u_*(kx - \omega t)$ solves

$$u_t = D\Delta u + f(u), \qquad \Delta = \partial_x^2 + \partial_y^2.$$



Co-moving frame $\zeta = kx - \omega t$**: stationary solution** u_* to

$$u_t = D\left(k^2 u_{\zeta\zeta} + u_{yy}\right) + \omega u_\zeta + f(u).$$

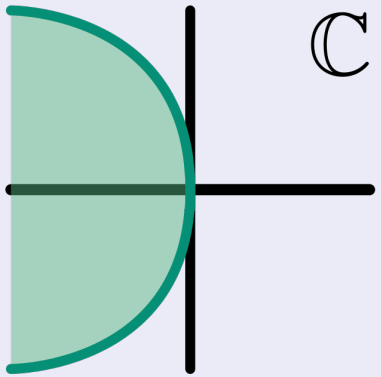
Linearization

$$\mathcal{L}_2 v = D\left(k^2 v_{\zeta\zeta} + v_{yy}\right) + \omega v_\zeta + f'(u_*(\zeta))v.$$

- $\mathcal{L}_2$ is ζ-**periodic** and y-**homogeneous** $\Longrightarrow \sigma(\mathcal{L}_2)$ parameterized over $S^1 \times \mathbb{R}$.
- **Translational invariance** $\Longrightarrow 0 \in \sigma(\mathcal{L}_2)$.

Planar wave trains

Weak spectral stability:



0 **simple** eigenvalue of $\mathcal{L}_1 = Dk^2\partial_{\zeta\zeta} + \omega\partial_\zeta + f'(u_*)$ in $L^2_{\mathrm{per}}([0,1])$;

Inherit weak spectral stability from **1D-wave train**? **Yes** (about 0) iff

$$\langle u_{\mathrm{ad}}, Du_*'\rangle_{L^2_{\mathrm{per}}} \langle u_{\mathrm{ad}}, u_*'\rangle_{L^2_{\mathrm{per}}} > 0,$$

where u_{ad} spans $\ker(\mathcal{L}_1^*)$ in $L^2_{\mathrm{per}}([0,1])$.

Planar wave trains

Weak spectral stability $\Longrightarrow$ nonlinear stability

Initial data:

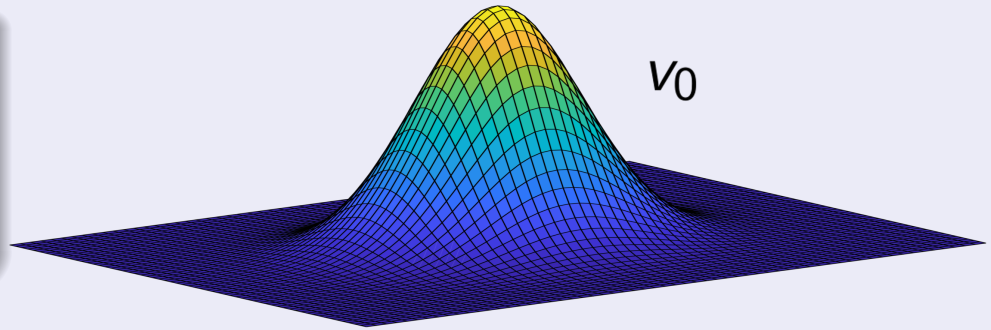
$$u(\zeta, y, 0) = u_*(\zeta) + v_0(\zeta, y).$$

Localized perturbations [dR, Sandstede '17]

$$\sup_{\zeta, y \in \mathbb{R}} \left(\|v_0(\zeta, y)\| + \|\partial_\zeta v_0(\zeta, y)\| \right) e^{\frac{\zeta^2 + y^2}{M}} \text{ small}$$

$$\Longrightarrow \|u(\zeta, y, t) - u_*(\zeta)\| \leq C \frac{e^{-\frac{|\zeta + \alpha t|^2 + y^2}{M(1+t)}}}{1+t}.$$

- **Diffusive decay**;
- α **group velocity**;
- **Faster** decay wrt 1D.



Planar wave trains

Weak spectral stability $\Longrightarrow$ nonlinear stability

Initial data:

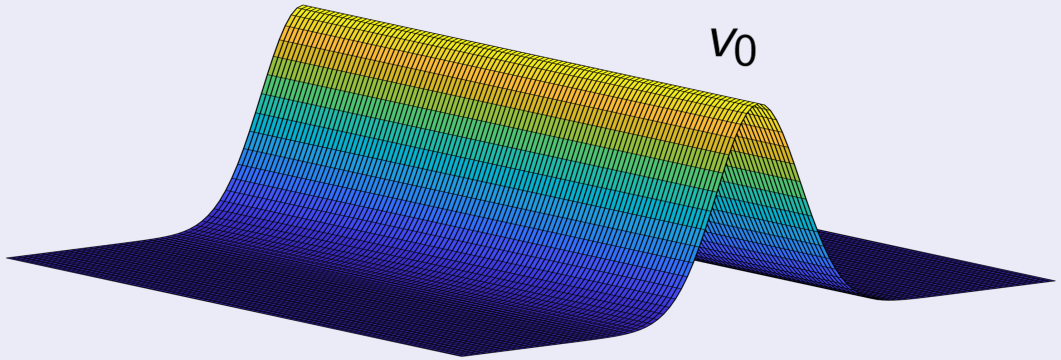
$$u(\zeta, y, 0) = u_*(\zeta) + v_0(\zeta, y).$$

Nonlocalized perturbations [dR, Sandstede '17]

$$\sup_{\zeta, y \in \mathbb{R}} \left(\|v_0(\zeta, y)\| + \|\partial_\zeta v_0(\zeta, y)\| \right) e^{\frac{|\beta\zeta + \gamma y|^2}{M}} \text{ small}$$

$$\Longrightarrow \|u(\zeta, y, t) - u_*(\zeta)\| \leq C \frac{e^{-\frac{|\beta\zeta + \alpha\beta t + \gamma y|^2}{M(1+t)}}}{\sqrt{1+t}},$$

with $(\beta, \gamma) \in \mathbb{R}^2$ unit vector.



Planar wave trains

Remarks

- **Nonlocalization** in **any** spatial direction in $\mathbb{R}^2$.
- Nonlocalization does **not originate** from phase modulation.
- **Trade-off** between localization and decay.

Intuition

- Diffusion smoothes out in **all** spatial directions.
- **Extra** spatial dimension **improves decay**.

Planar wave trains

Proof

- **Pointwise bounds** to exploit spatial structure.
  $\longrightarrow$ developed by Zumbrun et al (see [Zumbrun, Howard '98]).
- **Modulation Ansatz** to account for **translational invariance**.

Challenge: modulation Ansatz requires L^2-**damping estimate**.
$\longrightarrow$ **prohibits** nonlocalization.

Resolution

- **Alternate** between modulated and nonmodulated coordinates.
- **Control derivatives** through nonmodulated Ansatz.
  $\longrightarrow$ circumvent damping estimate.
- **Close nonlinear iteration** using modulated Ansatz.

Summary

Nonlinear stability analysis of wave trains from 1D to 2D:

- **Sign criterion:** 2D-stability induced by 1D-stability?
- Extra spatial dimension yields **larger class** of **nonlocal** perturbations.
  $\longrightarrow$ Nonlocality is structurally different (no phase modulation).

Future directions:

- **Different** (nonlocal) perturbations (algebraic weights)
- **Higher** spatial dimensions
- **Spirals**
- Stability against nonlocal perturbations in **1D**?

Questions

Thank you for the attention!

- B. de Rijk, B. Sandstede. Diffusive stability against nonlocalized perturbations of planar wave trains in reaction-diffusion systems, *preprint online*