# The Effect of Different Velocities on Global Existence and Stability in Reaction-Diffusion-Advection Systems

## MS20: Existence and Stability of Nonlinear Waves

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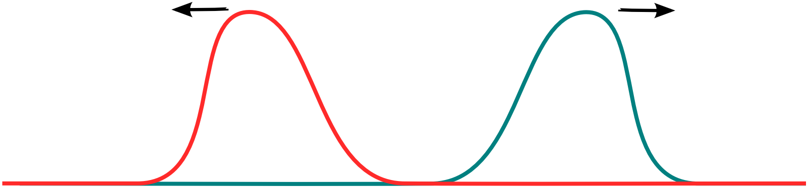


*Joint work with Guido Schneider

# Introduction

## Nonlinearly coupled reaction-diffusion-advection equations

$$u_t = \underbrace{D\Delta u}_{\textit{diffusion}} + \underbrace{\sum_{i=1}^{d} C_i u_{x_i}}_{\textit{drift}} + \underbrace{F(u)}_{\textit{reaction}}, \qquad u(t)\colon \mathbb{R}^d \to \mathbb{R}^n. \qquad (*)$$

- $D, C_i$ diagonal matrices with $D \geq 0$
- **Nonlinear reaction:** $F(0), \nabla F(0) = 0 \implies$ **no spectral gap!**

Classical problem: **global behavior of small solutions** to $(*)$.

## $(*)$ arises in...

1) ...various applications

   $\rightsquigarrow$ e.g. mass-action kinetics yield purely nonlinear reactions

2) ...nonlinear stability of wave trains

   $\rightsquigarrow$ no spectral gap due to translational invariance

# Introduction

**Example: autocatalytic reaction $A + 2B \to 3B$**

$$a_t = d_1 \Delta a + c a_x - ab^2$$
$$b_t = d_2 \Delta b + ab^2$$

Reactant drifts wrt autocatalyst with speed $c$.

# Formulate as Cauchy problem

Take $X = L^1(\mathbb{R}^d, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^n)$ and

$$A = D\Delta + \sum_{i=1}^{d} C_i \partial_{x_i}.$$

$Y$ space of initial conditions with $Y \hookrightarrow X$ dense.

**Cauchy problem for small initial data**

$$\begin{cases} \partial_t u = Au + F(u), \\ u(0) \in B_Y(\varepsilon) := \{u \in Y \mid \|u\|_Y < \varepsilon\}, \\ 0 < \varepsilon \ll 1. \end{cases} \tag{CP}$$

- **Local existence** via standard contraction mapping principle.
- **Global behavior is subtle as** $\sup \mathrm{Re}(\sigma(A)) = 0$.
  $\rightsquigarrow$ Global existence in $X$? Temporal decay rates?
  $\rightsquigarrow$ Nonlinear stability of $u \equiv 0$ in (CP)?

# Effect of the nonlinearity

Take $X = [L^1 \cap L^\infty](\mathbb{R}^d, \mathbb{R}^n)$ and $A = D\Delta + \sum_{i=1}^d C_i \partial_{x_i}$.

$$\begin{cases} \partial_t u = Au + F(u), \\ u(0) \in B_Y(\varepsilon), \\ 0 < \varepsilon \ll 1. \end{cases} \tag{CP}$$

**Fujita exponent:** $\varphi(d) := 1 + \frac{2}{d}$. **Assume:** $|F(u)| \lesssim |u|^p$.

$p > \varphi(d) \Longrightarrow \exists\, Y \subset X$ dense st all solutions to (CP) are global [Fujita '66]

$p \leq \varphi(d) \rightsquigarrow$ global behavior might depend on structure $F$

For instance, when $d = n = 1$, $\exists\, Y \subset X$ dense st

- $F(u) = -u^3 \rightsquigarrow$ (CP) always has a global solution (comp. principle)
- $F(u) = u^3 \rightsquigarrow \exists$ solutions to (CP) blowing up in $X$ [Hayakawa '73]

# Effect of velocities

Take $X = [L^1 \cap L^\infty](\mathbb{R}^d, \mathbb{R}^n)$ and $A = D\Delta + \sum_{i=1}^d C_i \partial_{x_i}$.

$$\begin{cases} \partial_t u = Au + F(u), \\ u(0) \in B_Y(\varepsilon), \\ 0 < \varepsilon \ll 1. \end{cases} \tag{CP}$$

**$p \leq \varphi(d) \rightsquigarrow$ global behavior might depend on velocities!**

For instance, when $d = 1$, $n = 2$ and

$$F(u) = \begin{pmatrix} u_1^{p_1} u_2^{q_1} \\ u_1^{p_2} u_2^{q_2} \end{pmatrix}, \qquad C_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

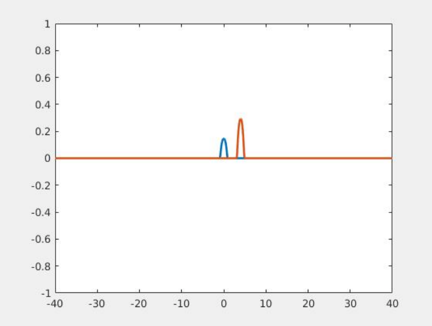
with $c_i \in \mathbb{R}$, $p_i, q_i \in \mathbb{Z}_{\geq 1}$ st $2 \leq p_i + q_i \leq 3 = \varphi(1)$. $\exists\, Y \subset X$ dense st

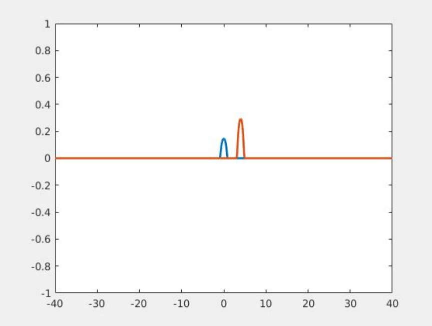
- $c_1 = c_2 \rightsquigarrow \exists$ solutions to (CP) blowing up in $X$ [Escobedo, Levine '95]
- $c_1 \neq c_2 \rightsquigarrow$ all solutions to (CP) are global [BdR, Schneider]

# Effect of velocities

**Take-home message**

For $p \leq \varphi(d)$ and $n \geq 2$ **differences in velocities can be decisive for the global dynamics**! (as is the nonlinearity)




$$z_t = z_{xx} + zw$$
$$w_t = w_{xx} + zw$$

$$z_t = z_{xx} + z_x + zw$$
$$w_t = w_{xx} + zw$$

# Formal analysis

## Temporal estimates

Solutions in $X = [L^1 \cap L^\infty](\mathbb{R}, \mathbb{R}^n)$ to

$$\begin{aligned} z_t &= d_1 z_{xx} + c_1 z_x \\ w_t &= d_2 w_{xx} + c_2 w_x \end{aligned} \tag{LIN}$$

decay as $\|z\|_\infty, \|w\|_\infty \sim t^{-1/2}$. So, if $p + q \leq 3$, then

$$\|z^p w^q\|_\infty \sim t^{-(p+q)/2} \geq t^{-3/2} \sim \|w^3\|_\infty.$$

**Not enough to close nonlinear iteration!**

# Formal analysis

## Spatio-temporal estimates

Assume solutions to

$$\begin{aligned} z_t &= d_1 z_{xx} + c_1 z_x \\ w_t &= d_2 w_{xx} + c_2 w_x \end{aligned} \qquad \text{(LIN)}$$

are drifting Gaussians: $z(x,t) \sim \frac{e^{-\frac{(x+c_1 t)^2}{4d_1 t}}}{\sqrt{t}}, \quad w(x,t) \sim \frac{e^{-\frac{(x+c_2 t)^2}{4d_2 t}}}{\sqrt{t}}.$

Then, for $p, q \geq 1$ we have

$$z^p w^q \sim \frac{e^{-p\frac{(x+c_1 t)^2}{4d_1 t} - q\frac{(x+c_2 t)^2}{4d_2 t}}}{t^{(p+q)/2}} \leq \frac{e^{-\frac{(c_1-c_2)^2 t}{4(d_1+d_2)}}}{t^{(p+q)/2}}.$$

Expect **exponential decay for mix-terms** if $c_1 \neq c_2$!
$\rightsquigarrow$ **enough to close nonlinear iteration** even if $p + q \leq 3$!

# Techniques

### How to make this rigorous?

Pointwise estimates (developed by Zumbrun and Howard for viscous shock waves)

# Global existence and decay via pointwise estimates

## Theorem [BdR, Schneider]

Let $Y = \{u \in C^1(\mathbb{R}, \mathbb{R}^2) : \sup_{x \in \mathbb{R}} |u(x) e^{x^2/M}| < \infty\}$, $M > 0$ and $c_i \in \mathbb{R}$ with $c_1 \neq c_2$. Suppose $f_i, g_i \in C^3(\mathbb{R}^2, \mathbb{R})$ satisfy

$$|f_i(z,w)| \lesssim \left(|z|^4 + |z||w| + |w|^4\right), \quad |g_i(z,w)| \lesssim \left(|z|^3 + |z||w| + |w|^3\right).$$

Then, $\forall\, \delta > 0 \;\; \exists\, \varepsilon > 0 \;\; \forall\, u_0 \in B_Y(\varepsilon)$ **the solution** $u(t) = (z,w)(t)$ **to**

$$z_t = d_1 z_{xx} + c_1 z_x + f_1(z,w) + \left(g_1(z,w)\right)_x,$$
$$w_t = d_2 w_{xx} + c_2 w_x + f_2(z,w) + \left(g_2(z,w)\right)_x,$$

**with** $u(0) = u_0$ **exists globally and satisfies**

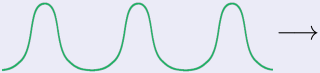
$$\|u(t)\|_\infty \leq \frac{\delta}{\sqrt{1+t}}, \qquad \|u(t)\|_1 \leq \delta, \qquad \text{for } t \geq 0.$$

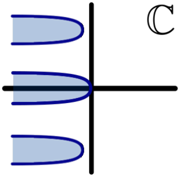
Also possible: $Y = \{u \in C^1(\mathbb{R}, \mathbb{R}^2) : \sup_{x \in \mathbb{R}} |u(x)(1+|x|)^r| < \infty\}$, $r \geq 3$.

# Application to wave trains at the Eckhaus boundary

# Crossing the Eckhaus boundary

Traveling **wave train** $u(x,t) = u_*(kx - \omega t)$ solves $u_t = Du_{xx} + f(u, \mu)$.

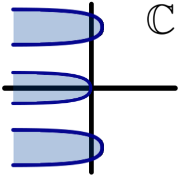




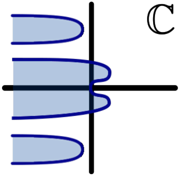

Weak spectral stability

alter $\mu$ ↙

alter $\mu$ ↘



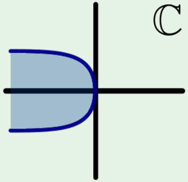

Hopf-unstable spectrum



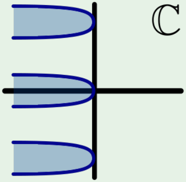

Sideband-unstable spectrum

# Dynamics at Eckhaus boundary

## Nonlinear stability at threshold?




**Sideband:** quartic touching!




**Hopf:** 3 critical modes!

**Nonlinear stability** proved in sideband case [Guillod et al. '18]

## Critical dynamics at Hopf destabilization (after mode filters)

$$z_t = d_1 z_{\xi\xi} + \partial_x \left(\alpha_1 z^2 + \alpha_2 w^2\right),$$

$$w_t = d_2 w_{\xi\xi} + \zeta w_\xi + \alpha_3 z w,$$

Idea: **exploit difference $\zeta$ in group velocities** between critical modes!

# Future directions

- **Application to RDA models with different velocities** (pipe flow, electro-RD systems)
- **Multiple spatial dimensions** (nonlocalized initial data?)
- **Nonlinear transport** (like in Burger's equation)
- **Beyond RDA systems**

# Questions

Thank you!

- B. de Rijk, G. Schneider. **Global existence and decay in nonlinearly coupled reaction-diffusion-advection equations with different velocities**, *in preparation*