The Effect of Different Velocities on Global Existence and Stability in Reaction-Diffusion-Advection Systems

MS20: Existence and Stability of Nonlinear Waves

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Introduction

Nonlinearly coupled reaction-diffusion-advection equations

\[ u_t = D \Delta u + \sum_{i=1}^{d} C_i u_x + F(u), \quad u(t): \mathbb{R}^d \to \mathbb{R}^n. \]  

- \( D, C_i \) diagonal matrices with \( D \geq 0 \)
- **Nonlinear reaction:** \( F(0), \nabla F(0) = 0 \implies \) no spectral gap!

Classical problem: **global behavior of small solutions** to (*)

(*) arises in...

1) ...various applications
   \( \rightsquigarrow \) e.g. mass-action kinetics yield purely nonlinear reactions

2) ...nonlinear stability of wave trains
   \( \rightsquigarrow \) no spectral gap due to translational invariance
Introduction

Example: autocatalytic reaction $A + 2B \rightarrow 3B$

\[
a_t = d_1 \Delta a + ca_x - ab^2
\]
\[
b_t = d_2 \Delta b + ab^2
\]

Reactant drifts wrt autocatalyst with speed $c$.
Formulate as Cauchy problem

Take $X = L^1(\mathbb{R}^d, \mathbb{R}^n) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^n)$ and

$$A = D\Delta + \sum_{i=1}^{d} C_i \partial x_i.$$ 

$Y$ space of initial conditions with $Y \hookrightarrow X$ dense.

Cauchy problem for **small** initial data

$$\begin{cases}
\partial_t u = Au + F(u), \\
u(0) \in B_Y(\varepsilon) := \{u \in Y \mid \|u\|_Y < \varepsilon\}, \\
0 < \varepsilon \ll 1.
\end{cases} \quad (CP)$$

- **Local existence** via standard contraction mapping principle.
- **Global behavior is subtle** as $\sup \text{Re}(\sigma(A)) = 0$.
  - $\Rightarrow$ Global existence in $X$? Temporal decay rates?
  - $\Rightarrow$ Nonlinear stability of $u \equiv 0$ in (CP)?
Effect of the nonlinearity

Take $X = [L^1 \cap L^\infty](\mathbb{R}^d, \mathbb{R}^n)$ and $A = D\Delta + \sum_{i=1}^d C_i \partial_{x_i}$.

\[
\begin{cases}
\partial_t u = Au + F(u), \\
u(0) \in B_Y(\varepsilon), \\
0 < \varepsilon \ll 1.
\end{cases} \tag{CP}
\]

**Fujita exponent:** $\varphi(d) := 1 + \frac{2}{d}$. \hspace{1cm} **Assume:** $|F(u)| \lesssim |u|^p$.

\[p > \varphi(d) \iff \exists \ Y \subset X \text{ dense st all solutions to (CP) are global} \text{ [Fujita '66]}\]

\[p \leq \varphi(d) \leadsto \text{global behavior might depend on structure } F\]

For instance, when $d = n = 1$, $\exists \ Y \subset X \text{ dense st}$

- $F(u) = -u^3 \leadsto (CP)$ always has a global solution (comp. principle)
- $F(u) = u^3 \leadsto \exists$ solutions to (CP) blowing up in $X$ [Hayakawa '73]
Effect of velocities

Take \( X = [L^1 \cap L^\infty](\mathbb{R}^d, \mathbb{R}^n) \) and \( A = D\Delta + \sum_{i=1}^d C_i \partial x_i \).

\[
\begin{cases}
\partial_t u = Au + F(u), \\
u(0) \in B_Y(\varepsilon), \\
0 < \varepsilon \ll 1.
\end{cases}
\]  

(\text{CP})

\( p \leq \varphi(d) \Leftrightarrow \) global behavior might depend on velocities!

For instance, when \( d = 1, n = 2 \) and

\[
F(u) = \begin{pmatrix}
u_1^{p_1} u_2^{q_1} \\
u_1^{p_2} u_2^{q_2}
\end{pmatrix}, \quad C_1 = \begin{pmatrix}c_1 & 0 \\
0 & c_2\end{pmatrix}
\]

with \( c_i \in \mathbb{R}, p_i, q_i \in \mathbb{Z}_{\geq 1} \) st \( 2 \leq p_i + q_i \leq 3 = \varphi(1) \). \( \exists \ Y \subset X \) dense \st

- \( c_1 = c_2 \Leftrightarrow \exists \) solutions to (CP) blowing up in \( X \) [Escobedo, Levine ‘95]
- \( c_1 \neq c_2 \Leftrightarrow \) all solutions to (CP) are global [BdR, Schneider]
Effect of velocities

**Take-home message**

For $p \leq \varphi(d)$ and $n \geq 2$ differences in velocities can be decisive for the global dynamics! (as is the nonlinearity)

\[
\begin{align*}
Z_t &= Z_{xx} + Zw \\
W_t &= W_{xx} + Zw
\end{align*}
\]

\[
\begin{align*}
Z_t &= Z_{xx} + Z_x + Zw \\
W_t &= W_{xx} + Zw
\end{align*}
\]
Formal analysis

Temporal estimates

Solutions in $X = [L^1 \cap L^\infty](\mathbb{R}, \mathbb{R}^n)$ to

$$
\begin{align*}
    z_t &= d_1 z_{xx} + c_1 z_x \\
    w_t &= d_2 w_{xx} + c_2 w_x
\end{align*}
$$

(LIN)

decay as $\|z\|_\infty, \|w\|_\infty \sim t^{-1/2}$. So, if $p + q \leq 3$, then

$$
\|z^p w^q\|_\infty \sim t^{-(p+q)/2} \geq t^{-3/2} \sim \|w^3\|_\infty.
$$

Not enough to close nonlinear iteration!
Spatio-temporal estimates

Assume solutions to

\[ z_t = d_1 z_{xx} + c_1 z_x \]
\[ w_t = d_2 w_{xx} + c_2 w_x \]

are drifting Gaussians:

\[ z(x, t) \sim e^{-\frac{(x+c_1 t)^2}{4d_1 t}} \frac{1}{\sqrt{t}}, \quad w(x, t) \sim e^{-\frac{(x+c_2 t)^2}{4d_2 t}} \frac{1}{\sqrt{t}}. \]

Then, for \( p, q \geq 1 \) we have

\[ z^p w^q \sim e^{-p \frac{(x+c_1 t)^2}{4d_1 t} - q \frac{(x+c_2 t)^2}{4d_2 t}} \frac{1}{t(p+q)/2} \leq e^{-\frac{(c_1 - c_2)^2 t}{4(d_1 + d_2)}} \frac{1}{t(p+q)/2}. \]

Expect exponential decay for mix-terms if \( c_1 \neq c_2 \),

\( \rightsquigarrow \text{enough to close nonlinear iteration} \) even if \( p + q \leq 3 \).
Techniques

How to make this rigorous?

Pointwise estimates (developed by Zumbrun and Howard for viscous shock waves)
Global existence and decay via pointwise estimates

Theorem [BdR, Schneider]

Let \( Y = \{ u \in C^1(\mathbb{R}, \mathbb{R}^2) : \sup_{x \in \mathbb{R}} |u(x)e^{x^2/M}| < \infty \}, \ M > 0 \) and \( c_i \in \mathbb{R} \) with \( c_1 \neq c_2 \). Suppose \( f_i, g_i \in C^3(\mathbb{R}^2, \mathbb{R}) \) satisfy

\[
|f_i(z, w)| \lesssim \left( |z|^4 + |z||w| + |w|^4 \right), \quad |g_i(z, w)| \lesssim \left( |z|^3 + |z||w| + |w|^3 \right).
\]

Then, \( \forall \delta > 0 \ \exists \ \varepsilon > 0 \ \forall \ u_0 \in B_Y(\varepsilon) \) the solution \( u(t) = (z, w)(t) \) to

\[
\begin{align*}
z_t &= d_1 z_{xx} + c_1 z_x + f_1(z, w) + (g_1(z, w))_x, \\
w_t &= d_2 w_{xx} + c_2 w_x + f_2(z, w) + (g_2(z, w))_x,
\end{align*}
\]

with \( u(0) = u_0 \) exists globally and satisfies

\[
\|u(t)\|_{\infty} \leq \frac{\delta}{\sqrt{1 + t}}, \quad \|u(t)\|_1 \leq \delta, \quad \text{for } t \geq 0.
\]

Also possible: \( Y = \{ u \in C^1(\mathbb{R}, \mathbb{R}^2) : \sup_{x \in \mathbb{R}} |u(x)(1 + |x|)^r| < \infty \}, \ r \geq 3. \)
Application to wave trains at the Eckhaus boundary
Crossing the Eckhaus boundary

Traveling wave train \( u(x, t) = u_*(kx - \omega t) \) solves \( u_t = Du_{xx} + f(u, \mu) \).

Weak spectral stability

Hopf-unstable spectrum

Sideband-unstable spectrum

 alteration of \( \mu \)
Dynamics at Eckhaus boundary

Nonlinear stability at threshold?

\[ \mathcal{C} \]

Sideband: quartic touching!

\[ \mathcal{C} \]

Hopf: 3 critical modes!

Nonlinear stability proved in sideband case [Guillod et al. ‘18]

Critical dynamics at Hopf destabilization (after mode filters)

\[
\begin{align*}
    z_t &= d_1 z_{\xi\xi} + \partial_x \left( \alpha_1 z^2 + \alpha_2 w^2 \right), \\
    w_t &= d_2 w_{\xi\xi} + \zeta w_{\xi} + \alpha_3 zw,
\end{align*}
\]

Idea: exploit difference \( \zeta \) in group velocities between critical modes!
Future directions

- Application to RDA models with different velocities (pipe flow, electro-RD systems)
- Multiple spatial dimensions (nonlocalized initial data?)
- Nonlinear transport (like in Burger’s equation)
- Beyond RDA systems
Questions

Thank you!

B. de Rijk, G. Schneider. Global existence and decay in nonlinearly coupled reaction-diffusion-advection equations with different velocities, in preparation