# Spectra and stability of spatially periodic pulse patterns II: the critical spectral curve

Björn de Rijk*

**Abstract**

In the stability analysis of pulse patterns in singularly perturbed reaction-diffusion systems, the scale separation is often exploited to reduce complexity. There are various methods to decompose the spectrum of the linearization about the pattern into slow and fast pieces and thus gain control over the spectrum in the asymptotic limit. These reduction methods have been developed in prototype models such as the FitzHugh-Nagumo, Gray-Scott and Gierer-Meinhardt equations, which are of slowly linear nature, i.e. the slow dynamics away from the pulses are driven by linear equations. Recently, these methods were extended to homoclinic pulses in a general, slowly nonlinear class of reaction-diffusion systems. Yet, a straightforward extension to periodic pulse solutions is impossible due to two major obstructions. First, the reduction methods that lead to asymptotic spectral control for the homoclinic pulses, fail for periodic patterns in slowly nonlinear systems. Second, there is a curve of spectrum attached to the origin that shrinks to the origin in the asymptotic limit, whereas for homoclinic pulses only a simple eigenvalue resides at the origin. Therefore, controlling the spectrum in the asymptotic limit is insufficient to decide upon stability. The first obstruction has been addressed in the companion paper [8]. In this paper we focus on the second obstruction: we obtain leading-order control over the critical spectral curve attached to the origin. The proof relies on Lin's method and utilizes exponential trichotomies. Our results yield explicit conditions for nonlinear stability and instability. Moreover, we gain insight into destabilization mechanisms of periodic pulse solutions.

**Keywords.** Spectral stability, reaction-diffusion systems, singular perturbations, periodic pulse patterns, Lin's method

## 1 Introduction

Reaction-diffusion systems with a strong spatial scale separation exhibit a large variety of patterns and have attracted much interest as a (simplified) model describing dynamical processes. For instance, they have been employed to model the propagation of nerve impulses through axons [24], the formation of spots and stripes on animal skin [33], the development of vegetation patterns [29] and the dynamics of flame fronts arising in combustion theory [52]. Nowadays, singularly perturbed, semi-linear reaction-diffusion systems on the line of the form

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H(u, v, \varepsilon), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u, v, \varepsilon), \end{aligned} \qquad u(\check{x}, t) \in \mathbb{R}^m, v(\check{x}, t) \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is asymptotically small and $D_{1,2}$ are non-negative diagonal matrices, serve as a paradigmatic class for the study of patterns. The scale separation in (1.1) induces a slow-fast decomposition in both the existence and stability analyses, which reduces complexity.

In the existence analysis, patterns can be obtained by concatenating orbit segments of slow and fast reduced systems, which arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of (1.1). The obtained patterns exhibit spatially localized fronts and pulses, while they vary slowly in between those localized interfaces – see Figure 1.

*Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

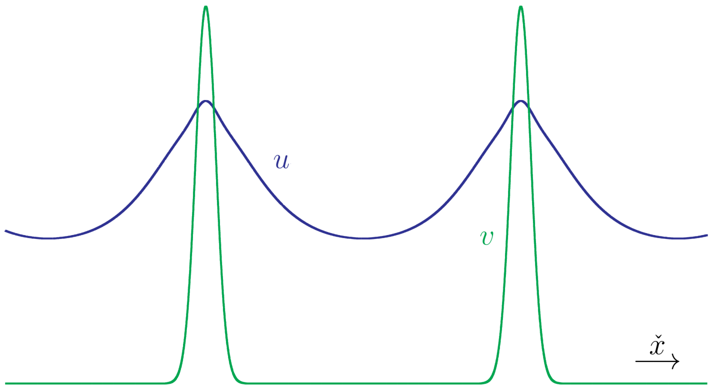



Figure 1: A periodic pulse solution in a reaction-diffusion system with two components. The $v$-component exhibits localized pulses and the $u$-component varies slowly.

In the stability analysis, the slow-fast decomposition manifests itself through a complex-analytic determinant-type function: *the Evans function* [1, 3, 16, 19], which vanishes on the spectrum of the linearization of (1.1) about the pattern and thus is a tool to locate the spectrum. In many specific models [1, 10, 15, 18, 39] it has been shown using geometric methods that the Evans function $\mathcal{E}_\varepsilon$ factorizes in accordance with the scale separation

$$\mathcal{E}_\varepsilon = \mathcal{E}_{s,\varepsilon} \cdot \mathcal{E}_{f,\varepsilon}, \tag{1.2}$$

into a slow and a fast component. Taking the limit $\varepsilon \to 0$ in (1.2) leads to an analytic reduced Evans function

$$\mathcal{E}_0 = \mathcal{E}_{s,0} \cdot \mathcal{E}_{f,0}, \tag{1.3}$$

whose factors can be derived explicitly through slow and fast *reduced* eigenvalue problems, which are *lower-dimensional* and arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of the full eigenvalue problem. The roots of $\mathcal{E}_0$ approximate those of $\mathcal{E}_\varepsilon$ via winding number arguments yielding control over the spectrum in the limit $\varepsilon \to 0$ – see Figure 2.

The geometric factorization procedure of the Evans function has been developed in the context of prototype models such as the Gray-Scott [10], Gierer-Meinhardt [9, 47] and FitzHugh-Nagumo equation [1, 15]. These models are of *slowly linear* nature, in the sense that the dynamics of the slow $u$-components in between localized fast pulses or fronts are driven by linear equations. In the context of the periodic pulse solution shown in Figure 1, slow linearity entails that the dynamics of (1.1) in the rest state $v = 0$ is linear, i.e. the coupling term $H(u, 0, \varepsilon)$ in (1.1) is linear. The geometric factorization procedure has been generalized in [14] to homoclinic pulse solutions in a general class of *slowly nonlinear* reaction-diffusion systems with two components. Earlier, the stability of fronts was studied in a specific slowly nonlinear model in [11].

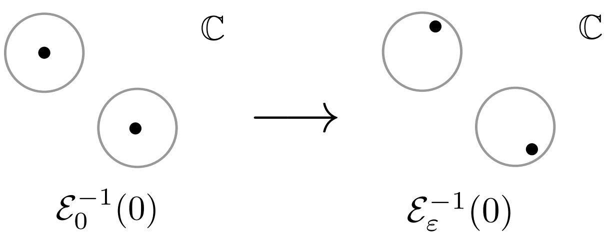
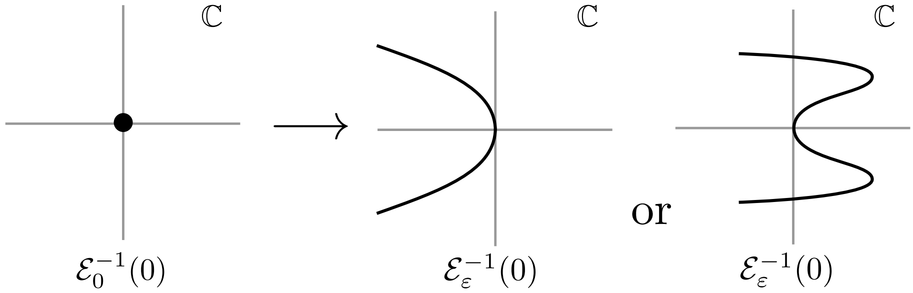
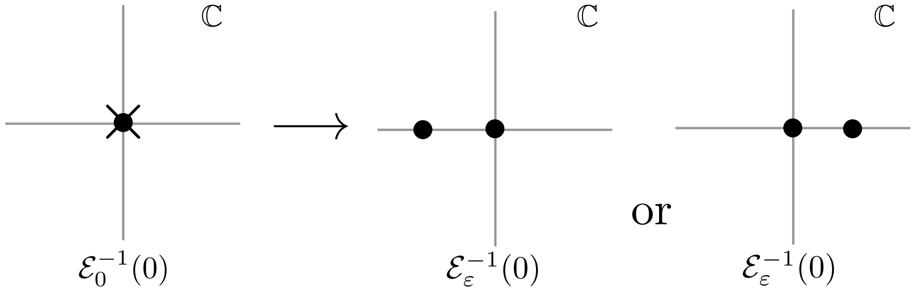



Figure 2: Spectral approximation using winding number arguments: the roots of $\mathcal{E}_0$ approximate those of $\mathcal{E}_\varepsilon$.




(a) Essential spectrum attached to the origin associated with periodic structures




(b) Point spectrum associated with fast pulses in the FitzHugh-Nagumo equations

Figure 3: Controlling the spectrum in the limit $\varepsilon \to 0$ is insufficient to determine the spectral configuration about the origin.

The introduction of a slow nonlinearity in (1.1) yields more than just additional technicalities. For instance, it is shown in [49] that, unlike known classical slowly linear examples such as the Gray-Scott and Gierer-Meinhardt models, Hopf bifurcations for homoclinic pulses can be supercritical in the slowly nonlinear regime. Such a bifurcation could even be the first step in a sequence of further bifurcations leading to complex (amplitude) dynamics of a standing solitary pulse – as suggested by simulations in [50]. These observations indicate that the dynamics of *periodic* pulse solutions in slowly nonlinear systems might be very rich.

This paper is devoted to *periodic* pulse solutions to a general class of *multi-component, slowly nonlinear* reaction-diffusion systems of the form (1.1), i.e. we allow for general dimensions $n, m \geq 1$ and a large class of nonlinearities $H$ and $G$ – see §2 for the precise details. The solutions under consideration are stationary and spatially symmetric. Moreover, they exhibit exponentially localized pulses in the fast $v$-components, but admit non-localized behavior in the slow $u$-components – see Figure 1.

Thus, our work can be considered as the extension from homoclinic to periodic structures within the general, slowly nonlinear class of systems in [14]. However, there are two major obstructions. First, the geometric factorization method of the Evans function fails for periodic structures in slowly nonlinear systems. Second, the spectrum is composed of curves parameterized over the unit circle $S^1$ by Floquet theory [19]. Due to translational invariance of the periodic pulse profile in space there is an entire curve of spectrum attached to the origin, whereas for the homoclinic pulse solutions in [14] there is only a simple eigenvalue residing at the origin. We will see that this critical spectral curve shrinks to the origin in the asymptotic limit $\varepsilon \to 0$. Therefore, controlling the spectrum in the limit $\varepsilon \to 0$ is insufficient to decide upon stability of periodic pulse solutions and a local higher-order analysis is required to determine the fine structure of the spectrum about the origin – see Figure 3a. This contrasts sharply with the homoclinic case in [14] where spectral and nonlinear stability – see [23] – follows if $\mathcal{E}_0^{-1}(0) \setminus \{0\}$ is contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$ and $\mathcal{E}_0'(0)$ is non-zero.

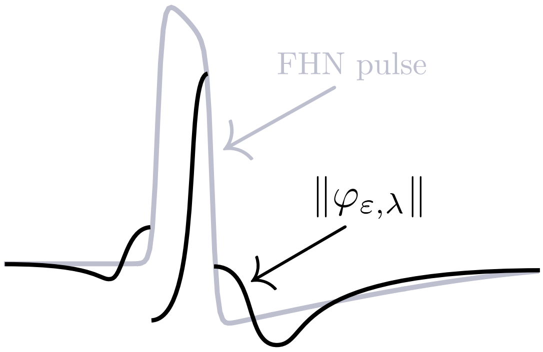



Figure 4: A piecewise continuous eigenfunction $\varphi_{\varepsilon,\lambda}$ with jumps occurring in the middle of the front and the back of the fast pulse to the FitzHugh-Nagumo equations.

The first obstruction has been addressed in [8], where a generalized analytic alternative to the geometric factorization procedure of the Evans function is presented that does work for periodic structures in the slowly nonlinear regime. In [8] one establishes the validity of both the decomposition (1.2) and its singular limit structure (1.3) by employing the Riccati transform. The factors $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$ can be determined explicitly in terms of lower-dimensional reduced eigenvalue problems. This leads to control over the spectrum in the limit $\varepsilon \to 0$.

In this follow-up paper we address the *second* obstruction: we obtain leading-order control over the critical spectral curve attached to the origin, i.e. we rigorously expand the curve in terms of the parameter $\varepsilon$. Our analysis is based on the methods developed in [5, 44] – we refer to Remark 1.1 for alternative methods. Note that the spectrum of the linearization is made up of point spectrum, consisting of isolated eigenvalues of finite multiplicity, and its complement, the essential spectrum [42]. In the stability analysis [5] of (fast) traveling pulses in the FitzHugh-Nagumo equations, 0 is a double root of the reduced Evans function $\mathcal{E}_0$, while the essential spectrum is confined to the open left half-plane. Consequently, there are two eigenvalues close to the origin. One of these eigenvalues resides at the origin due to translational invariance, while the position of the other eigenvalue with respect to the imaginary axis is decisive for stability. Yet, controlling the spectrum in the limit $\varepsilon \to 0$ is insufficient to determine its location – see Figure 3b. There are several approaches to attack this problem – see [25, 51]. In [5] one constructs using *Lin's method* [30, 41, 48] a piecewise continuous eigenfunction of the linearization for each prospective eigenvalue $\lambda$ near the origin. The eigenfunction admits exactly two jumps that occur in the middle of the front and the back of the pulse profile – see Figure 4. Finding eigenvalues reduces to identifying values of $\lambda$ for which these jumps vanish. Melnikov theory provides leading-order expressions for these jumps that can be solved for $\lambda$, which yields that the critical eigenvalue lies in the open left half-plane. This proves nonlinear stability for the traveling pulse.

In [44] the spectral configuration about the origin is determined for (long-wavelength) periodic wave trains that accompany homoclinic pulse solutions to general reaction-diffusion systems. The spectral curves associated to the wave trains shrink to the eigenvalues associated with the limiting homoclinic as the wavelength tends to infinity [20, 43]. Therefore, spectral stability of the homoclinic pulse is insufficient to decide upon stability of nearby periodic wave trains, because the translational eigenvalue of the homoclinic at the origin generates a critical curve of essential spectrum of the wave trains that touches the imaginary axis. One proceeds in [44] by constructing a piecewise continuous eigenfunction for every potential eigenvalue near the origin using Lin's method. The eigenfunction admits exactly one jump on a single periodicity interval, which can be expressed in terms of the eigenvalue $\lambda$, the Floquet multiplier $\gamma$ and the period $L$. Using Melnikov theory one obtains an expansion of the critical spectral curve $\lambda_L(\gamma)$ in terms of the period $L$.

The situations in [5] and [44] do not directly translate to our situation, since we consider periodics that do not lie in the vicinity of a homoclinic. Nevertheless, we adopt a similar approach: using Lin's method we obtain a piecewise continuous eigenfunction of the linearization for any potential eigenvalue $\lambda$ near the origin. In contrast to [44], we do not use the homoclinic limit structure as a framework for the construction of the eigenfunction; instead the singular limit structure serves as a backbone like in [5]. On the other hand, as in [44], Floquet theory yields boundary conditions for the eigenfunction on a single periodicity interval, whereas one requires in [5] that the eigenfunction is exponentially localized on the real line.

Our construction of the piecewise continuous eigenfunction yields a Lyapunov-Schmidt type reduction procedure: finding the critical spectral curve attached to the origin reduces to equating the jumps to zero. The Fredholm alternative allows us to find expressions for these jumps that can then be solved. Eventually, we obtain a leading-order expression for the critical spectral curve in terms of lower-dimensional, variational equations about the orbit segments that constitute the periodic pulse profile in the limit $\varepsilon \to 0$. Moreover, we find that the critical spectral curve is confined to the real axis and scales with $\varepsilon^2$.

Thus, we gain both control over the spectrum in the limit $\varepsilon \to 0$ through the Evans-function analysis in [8] and leading-order control over the critical spectral curve through the analysis in this paper. This leads to *explicit* criteria yielding nonlinear stability and instability of the periodic pulse solution in terms of simpler, lower-dimensional (eigenvalue) problems. We refer to [12] for calculations in an explicit slowly nonlinear model equation that illustrate this reduction.

Our spectral analysis simplifies in the cases in which either $n = 1$, $m = 1$, or both $n = m = 1$. Then, the associated reduced eigenvalue problems are of Sturm-Liouville type and well-understood. In the case $n = m = 1$, our results yield deep insights into destabilization mechanisms of periodic pulse solutions to (1.1). If a periodic pulse solution destabilizes, then generically one of the aforementioned stability criteria fails, whereas the others are still valid. The obtained spectral grip allows us to precisely identify the type of instability occurring depending on which one of these criteria fail. In addition, we establish that generic (primary) instabilities must be of sideband, Hopf or period doubling type, whereas in general reaction-diffusion systems also Turing and fold instabilities are robust for symmetric, spatially periodic patterns [38].

This paper is structured as follows. In §2 we introduce the class of reaction-diffusion systems under consideration. Moreover, we review and refine the existence result from [8], because the analysis in this paper requires sharper estimates on the error between the periodic pulse solutions and their singular limit profile. In §3 we present the main result of this paper, i.e. we provide the expansion of the critical spectral curve attached to the origin. Together with the spectral control through the Evans-function analysis in [8], our main result yields explicit conditions in terms of simpler, lower-dimensional systems for nonlinear stability and instability. In §4 we focus on destabilization mechanisms of periodic pulse solutions. In §5 we perform our actual spectral analysis and establish the leading-order geometry of the critical spectral curve using Lin's method. Finally, Appendix A contains some technical results on exponential di- and trichotomies, which are needed for our spectral analysis.

**Remark 1.1.** In [15, 47] one studies the stability of periodic structures in the FitzHugh-Nagumo and Gierer-Meinhardt models. As in our situation, an entire curve of eigenvalues is attached to the origin, which shrinks to the origin in the limit $\varepsilon \to 0$. The stability properties of the periodic pattern depend on the leading-order geometry of this critical spectral curve. Instead of Lin's method, one uses different methods in [15, 47] to determine the leading-order geometry. In the stability analysis [15] of wave trains in the FitzHugh-Nagumo equations one employs power series expansions of certain integral operators to control the asymptotic behavior of solutions to the eigenvalue problem. A careful matching procedure then gives two solvability conditions, which lead to the desired critical spectral curve. Together with an Evans-function analysis, this yields nonlinear stability of the periodic wave train.

Finally, in the stability analysis of periodic pulse solutions to the Gierer-Meinhardt equations in [47], one applies a Fredholm alternative to the eigenvalue problem, which leads to a solvability condition in terms of the eigenvalue $\lambda$, the Floquet multiplier $\gamma$ and the corresponding eigenfunction. By expanding the eigenfunction in terms of $\varepsilon$ and $\gamma$ in- and outside the pulse region separately, a leading-order expression for the critical spectral curve $\lambda_\varepsilon(\gamma)$ is obtained. By complementing the analysis about the origin with an Evans-function analysis, stability or instability of the periodic pulse pattern is established depending on system parameters. ■

**Remark 1.2.** This paper focuses on stationary, spatially symmetric, periodic pulse solutions to (1.1). Spatially symmetric solutions arise naturally when substituting a stationary wave Ansatz in (1.1), because the associated existence problem admits a reversible symmetry. Moreover, linearizing (1.1) about a stationary, spatially symmetric solution, yields a reversibly symmetric eigenvalue problem. These symmetries can be broken by adding *advection* terms to system (1.1) or by studying *traveling*-wave solutions to (1.1) instead of stationary ones. In some applications advection terms occur naturally, leading to reaction-advection-diffusion models [2, 29]. It is therefore an interesting and relevant question how symmetry breaking affects our analysis.

In general, the spectrum associated with periodic wave trains to reaction-advection-diffusion systems consists of continuous images of the unit circle $S^1$ [19]. The presence of symmetry yields degenerate spectrum: the image of $S^1$ covers each curve of spectrum twice. Thus, breaking the symmetry changes the structure of the spectrum fundamentally. Our expansion method of the critical spectral curve is based on the analyses in [5, 44] using Lin's method. Both in [5] and in [44] the eigenvalue problem does not admit a (reversible) symmetry, since one considers *traveling* waves. Therefore, we expect that the present expansion method remains valid in the non-symmetric case. Yet, we foresee that the outcome of the analysis will be different: we conjecture that the critical spectral curve is no longer confined to the real axis and scales with $\varepsilon$ instead of $\varepsilon^2$, since the $O(\varepsilon)$-terms in the expansion will no longer vanish due to parity arguments. Our hypothesis is further strengthened by the fact that the critical spectral curve associated with periodic traveling waves in the FitzHugh-Nagumo equations scales with $\varepsilon$ and is non-real – see [15]. ■

# 2 Setting

## 2.1 Class of slowly nonlinear reaction-diffusion systems

In this paper we consider the same general class of multi-component, slowly nonlinear reaction-diffusion systems as in the companion paper [8]. We shortly recap the derivation in [8, Section 2.1] of our model class from the general reaction-diffusion system (1.1). First, we allow for general dimensions $m, n \in \mathbb{Z}_{>0}$ in (1.1) and we require that the diagonal matrices $D_{1,2}$ are *positive*. Next, we write

$$H(u, v, \varepsilon) = H(u, 0, \varepsilon) + \tilde{H}_2(u, v, \varepsilon),$$

where $\tilde{H}_2(u, v, \varepsilon) := H(u, v, \varepsilon) - H(u, 0, \varepsilon)$, so that $\tilde{H}_2$ vanishes at $v = 0$. To sustain stable localized patterns in semi-strong interaction [37] in system (1.1), we allow $\tilde{H}_2(u, v, \varepsilon)$ to scale with $\varepsilon^{-1}$ and define

$$H_2(u, v) := \lim_{\varepsilon \to 0} \varepsilon \tilde{H}_2(u, v, \varepsilon).$$

Thus, we obtain

$$H(u, v, \varepsilon) = H_1(u, v, \varepsilon) + \varepsilon^{-1} H_2(u, v),$$

with $H_1(u, v, \varepsilon) := H(u, 0, \varepsilon) + [\tilde{H}_2(u, v, \varepsilon) - \varepsilon^{-1} H_2(u, v)]$. By construction $H_2(u, v)$ vanishes at $v = 0$. We emphasize that $H_2(u, v) \equiv 0$ is allowed in our analysis. However, in the case $n = 1$, this yields only unstable periodic pulse solutions – see [8, Remark 3.14]. Finally, we require that $G$ vanishes at $v = 0$.

We refer to [8, Remark 2.6] for an extensive discussion of this last condition. Thus, our model class is of the form

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H_1(u,v,\varepsilon) - \varepsilon^{-1} H_2(u,v), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u,v,\varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, \check{x} \in \mathbb{R}, \tag{2.1}$$

or, in the small spatial scale $x = \varepsilon^{-1}\check{x}$

$$\begin{aligned} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u,v,\varepsilon) - \varepsilon H_2(u,v), \\ v_t &= D_2 v_{xx} - G(u,v,\varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, x \in \mathbb{R}. \tag{2.2}$$

The aforementioned conditions read:

**(S1)** ***Conditions on the interaction and diffusion terms***
There exists open, connected sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ with $0 \in V$ and $0 \in I$ such that $H_1, G$ and $H_2$ are $C^3$ on their domains $U \times V \times I$ and $U \times V$, respectively. Moreover, we have $H_2(u,0) = 0$ and $G(u,0,\varepsilon) = 0$ for all $u \in U$ and $\varepsilon \in I$. Finally, $D_{1,2}$ are positive diagonal matrices.

**Remark 2.1.** If we have $n = 1$, we can without loss of generality assume $D_2 = 1$ in (2.1) by rescaling the spatial variable $\check{x}$. Similarly, in the case $m = 1$, we can without loss of generality assume $D_1 = 1$ by rescaling the parameter $\varepsilon$. ■

## 2.2 Existence of periodic pulse solutions

This paper is concerned with stationary, reversibly symmetric, periodic pulse solutions to the slowly nonlinear class (2.2) of reaction-diffusion equations. In this subsection, we outline the construction of such solutions as exhibited in [8, Section 2.3]. Moreover, we present a refinement of some of the existence results from [8], because the analysis in this paper requires sharper estimates on the error between the periodic pulse solutions and their singular limit profile.

Finding stationary solutions to (2.2) is equivalent to solving the singularly perturbed ordinary differential equation

$$\begin{aligned} D_1 u_x &= \varepsilon p, \\ p_x &= \varepsilon H_1(u,v,\varepsilon) + H_2(u,v), \\ D_2 v_x &= q, \\ q_x &= G(u,v,\varepsilon), \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^{2(m+n)}, \tag{2.3}$$

which is $R$-reversible, where $R \colon \mathbb{R}^{2(m+n)} \to \mathbb{R}^{2(m+n)}$ is the reflection in the space $p = q = 0$. In [8, Section 2.3] it is shown that periodic pulse solutions to (2.3) arise from a concatenation of solutions to a series of reduced subsystems of (2.3). If we set $\varepsilon = 0$ in (2.3), we obtain the *fast reduced system*

$$\begin{aligned} u_x &= 0, \\ p_x &= H_2(u,v), \\ D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^{2(m+n)}. \tag{2.4}$$

System (2.4) is governed by the family of $2n$-dimensional systems

$$\begin{aligned} D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (v,q) \in \mathbb{R}^{2n}, \tag{2.5}$$

parameterized over $u \in U$. Note that (2.5) is $R_f$-reversible, where $R_f\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the reflection in the space $q = 0$. We observe that *the slow manifold*

$$\mathcal{M} := \{(u, p, 0, 0) : u \in U, p \in \mathbb{R}^m\},$$

consists entirely of equilibria of (2.4) by assumption **(S1)**. We require $\mathcal{M}$ to be normally hyperbolic.

**(S2)** ***Normal hyperbolicity***
For each $u \in U$ the symmetric part $\mathrm{Re}(\mathcal{G}(u)) = \frac{1}{2}\left(\mathcal{G}(u) + \mathcal{G}(u)^T\right)$ of $\mathcal{G}(u) := \partial_v G(u, 0, 0)$ is positive definite.

When $\varepsilon > 0$, the manifold $\mathcal{M}$ consists no longer of equilibria, but remains invariant. In the spatial scale $\check{x} = \varepsilon x$, the flow restricted to $\mathcal{M}$ is to leading order governed by the *slow reduced system*

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= H_1(u, 0, 0), \end{aligned} \qquad (u, p) \in \mathbb{R}^{2m}. \tag{2.6}$$

System (2.6) is $R_s$-reversible, where $R_s\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is the reflection in the space $p = 0$.

The dynamics around the normally hyperbolic manifold $\mathcal{M}$ is captured by Fenichel's geometric singular perturbation theory [17]. In [8, Section 2.3] a singular periodic orbit is constructed by concatenating a pulse solution to the fast reduced system (2.4) with an orbit segment on the slow manifold $\mathcal{M}$ governed by the slow reduced system (2.6). Subsequently, Fenichel's theory is employed to establish an actual periodic pulse solution to (2.3) in the vicinity of the singular one.

The existence of a pulse solution in the fast reduced system (2.4) is guaranteed by the following assumption:

**(E1)** ***Existence of a pulse solution to the fast reduced system***
There exists $u_\diamond \in U$ such that (2.5) has for $u = u_\diamond$ a solution $\psi_\mathrm{h}(x, u_\diamond) = (v_\mathrm{h}(x, u_\diamond), q_\mathrm{h}(x, u_\diamond))$ homoclinic to 0. The stable manifold $W^s_{u_\diamond}(0)$ intersects the space $\ker(I - R_f)$ transversely in the point $\psi_\mathrm{h}(0, u_\diamond)$.

Since transverse intersections are robust under perturbations, assumptions **(S2)** and **(E1)** imply the existence of an open neighborhood $U_\mathrm{h} \subset U$ of $u_\diamond$ such that for every $u \in U_\mathrm{h}$ there exists a solution $\psi_\mathrm{h}(x, u)$ to (2.5), which is homoclinic to 0, such that $W^s_u(0) \pitchfork \ker(I - R_f) = \{\psi_\mathrm{h}(0, u)\}$. The homoclinics $\psi_\mathrm{h}(x, u)$ yield solutions

$$\phi_\mathrm{h}(x, u) := \left(u, \int_0^x H_2(u, v_\mathrm{h}(z, u))dz, v_\mathrm{h}(x, u), q_\mathrm{h}(x, u)\right), \quad u \in U_\mathrm{h},$$

to the fast reduced system (2.4), which are homoclinic to $\mathcal{M}$. The limit points $\lim_{x\to\pm\infty} \phi_\mathrm{h}(x, u)$ lead to the so-called take-off and touch-down manifolds $\mathcal{T}_\pm := \{(u, \pm\mathcal{J}(u)) : u \in U_\mathrm{h}\}$ on $\mathcal{M}$, where $\mathcal{J}\colon U_\mathrm{h} \to \mathbb{R}^m$ is defined by

$$\mathcal{J}(u) = \int_0^\infty H_2(u, v_\mathrm{h}(z, u))dz. \tag{2.7}$$

Since 0 is a hyperbolic saddle in (2.5), there exists constants $C, \mu_\mathrm{h} > 0$ such that

$$\|\phi_\mathrm{h}(\pm x, u) - (u, \pm\mathcal{J}(u), 0, 0)\| \le Ce^{-\mu_\mathrm{h} x}, \quad x \ge 0, u \in U_\mathrm{h}. \tag{2.8}$$

The manifolds $\mathcal{T}_\pm$ allow us to piece the pulse solutions $\phi_\mathrm{h}$ to orbit segments on $\mathcal{M}$. By reversibility of the slow and fast reduced systems, it holds $R_s[\mathcal{T}_+] = \mathcal{T}_-$. Therefore, to establish a connection between the take-off and touch-down manifolds $\mathcal{T}_\pm$, it is sufficient to find an orbit in (2.6) that starts on the touch-down manifold $\mathcal{T}_+$ and crosses $\ker(I - R_s)$ at some point. This is ensured by the following assumption.

**(E2)** ***Existence of connecting orbit in slow reduced system***
There exists a solution $\psi_s(\check{x}) = (u_s(\check{x}), p_s(\check{x}))$ to system (2.6) with initial condition $\psi_s(0) \in \mathcal{T}_+$ and $\psi_s(\ell_0) \in \ker(I - R_s)$ for some $\ell_0 > 0$. Moreover, let $\Phi_s(\check{x}, \check{y})$ be the evolution operator of the associated variational equation

$$\varphi_{\check{x}} = \mathcal{A}_s(\check{x})\varphi, \quad \varphi \in \mathbb{R}^{2m}, \tag{2.9}$$

with

$$\mathcal{A}_s(\check{x}) := \begin{pmatrix} 0 & D_1^{-1} \\ \partial_u H_1(u_s(\check{x}), 0, 0) & 0 \end{pmatrix}.$$

Denote $u_0 := u_s(0)$, $H_1(u_0, 0, 0) = (h_1, \ldots, h_m)$ and for $i, j \in \{1, \ldots, m\}$ by $A_{ij}$ the $(m{\times}m)$-submatrix of

$$\Phi_s(\ell_0, 0)\begin{pmatrix} I \\ \partial_u J(u_0) \end{pmatrix},$$

containing rows $\{i, m+1, \ldots, 2m\} \setminus \{m+j\}$. There exists $i_* \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{m} (-1)^j h_j \det(A_{i_* j}) \neq 0. \tag{2.10}$$

We will refer to (2.9) as the *slow variational equation*. By concatenating the orbits of $\psi_s$ and $\phi_h$, we obtain the *singular periodic pulse*

$$\phi_{p,0} := \{(\psi_s(\check{x}), 0) : \check{x} \in (0, 2\ell_0)\} \cup \{\phi_h(x, u_0) : x \in \mathbb{R}\} \subset \mathbb{R}^{2(m+n)}. \tag{2.11}$$

We emphasize that $\phi_{p,0}$ is smooth, except at the two *corners* $(u_0, \pm\mathcal{J}(u_0), 0, 0) = (u_s(0), \pm p_s(0), 0, 0)$. Robustness of the structure (2.11) at the corners is ensured by the technical (transversality) condition (2.10) in assumption **(E2)** – we refer to [8, Section 2.3] for a more extensive discussion.

In [8, Theorem 2.11], Fenichel's theory is employed to construct a periodic pulse solution to (2.3) close to the singular concatenation (2.11) for $0 < \varepsilon \ll 1$. We provide a more refined result below, since we require sharper estimates for the analysis in this paper.

**Theorem 2.2.** *Assume **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Then, there exists constants $C, \mu_0, \varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a solution $\phi_{p,\varepsilon}(x)$ to* (2.3) *satisfying the following assertions:*

1. ***Periodicity***
*$\phi_{p,\varepsilon}$ is $2L_\varepsilon$-periodic, where*

$$|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon. \tag{2.12}$$

2. ***Reversibility***
*We have $\phi_{p,\varepsilon}(x) = R\phi_{p,\varepsilon}(-x)$ for $x \in \mathbb{R}$.*

3. ***Singular limit***
*Define for $\theta \geq \mu_0^{-1}$ the quantity $\Xi_\theta(\varepsilon) := -\theta \log(\varepsilon)$. The solution $\phi_{p,\varepsilon}$ approximates the pulse as*

$$\left\|\phi_{p,\varepsilon}(x) - \phi_h(x, u_0)\right\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon), \Xi_\theta(\varepsilon)], \tag{2.13}$$

*and it approximates the orbit segment on the slow manifold as*

$$\left\|\phi_{p,\varepsilon}(x) - (\psi_s(\varepsilon x), 0)\right\| \leq C\varepsilon, \quad x \in [\Xi_\theta(\varepsilon), 2L_\varepsilon - \Xi_\theta(\varepsilon)]. \tag{2.14}$$

4. ***Exponential convergence to slow manifold***
*We have the estimate*

$$d(\phi_{\mathrm{p},\varepsilon}(x), \mathcal{M}) \leq Ce^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}.$$

**Proof.** In the following, we denote by $C > 0$ a constant, which is independent of $\varepsilon$. Moreover, we choose a $2m$-dimensional compact submanifold $\mathcal{M}_0$ of $\mathcal{M}$ that contains the orthogonal projection of the singular orbit (2.11) on $\mathcal{M}$.

Besides the estimates (2.12), (2.13) and (2.14) the result follows directly from [8, Theorem 2.11]. To prove the remaining estimates, we put system (2.3) into its *Fenichel normal form* – see [26, Proposition 1] – in an $\varepsilon$-independent neighborhood $\mathcal{D} \subset \mathbb{R}^{2(m+n)}$ of $\mathcal{M}_0$. For $0 \leq \varepsilon \ll 1$, there exists a $C^1$-change of coordinates $\Psi_\varepsilon \colon \mathcal{D} \to \mathbb{R}^{2(m+n)}$, depending $C^1$-smoothly on $\varepsilon$, in which the flow of (2.3) is given by

$$\begin{aligned} a_x &= A(a,b,c,\varepsilon)a, \\ b_x &= B(a,b,c,\varepsilon)b, \qquad\qquad a, b \in \mathbb{R}^n, c \in \mathbb{R}^{2m}, \\ c_x &= \varepsilon K(c,\varepsilon) + H(a,b,c,\varepsilon)(a \otimes b), \end{aligned} \tag{2.15}$$

where the functions $A, B, K$ and $H$ are $C^1$ in their arguments, $K$ maps to $\mathbb{R}^{2m}$, $A$ and $B$ map to the square matrices of order $n$ and $H$ maps to tensors of appropriate rank. Moreover, there exists $\Delta > 0$ and an open and bounded set $U_F \subset \mathbb{R}^{2m}$ such that the image $\Psi_\varepsilon(\mathcal{D})$ contains the compact box

$$\mathcal{B} := \left\{(a,b,c) : \|a\|, \|b\| \leq \Delta, c \in \overline{U_F}\right\}.$$

In addition, there exists $\mu_0 > 0$, independent of $\varepsilon$, such that

$$\mathrm{Re}(\sigma(A(a,b,c,\varepsilon))) \leq -\mu_0, \quad \mathrm{Re}(\sigma(B(a,b,c,\varepsilon))) \geq \mu_0, \tag{2.16}$$

and

$$\|H(a,b,c,\varepsilon)(a \otimes b)\| \leq C\|a\|\|b\|, \tag{2.17}$$

for all $(a,b,c) \in \Psi_\varepsilon(\mathcal{D})$ and $0 \leq \varepsilon \ll 1$. Note that system

$$c_{\check{x}} = K(c,0), \quad c \in \mathbb{R}^{2m},$$

is equivalent to the slow reduced system (2.6).

We recall two facts from the proof of [8, Theorem 2.11]. First, there exists $x_0 > 0$, which is independent of $\varepsilon$, such that the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x)$ is approximated as

$$\left\|\phi_{\mathrm{p},\varepsilon}(x) - \phi_{\mathrm{h}}(x, u_0)\right\| \leq C\varepsilon, \tag{2.18}$$

for $x \in [-x_0, 0]$. Second, the solution $\phi_{\mathrm{p},\varepsilon}(x)$ is contained in the neighborhood $\mathcal{D}$ for $x \in [-L_\varepsilon, -x_0]$.

We express $\phi_{\mathrm{p},\varepsilon}(x)$ in Fenichel coordinates as $\tilde{\phi}_{\mathrm{p},\varepsilon}(x) = (a_{\mathrm{p},\varepsilon}(x), b_{\mathrm{p},\varepsilon}(x), c_{\mathrm{p},\varepsilon}(x)) = \Psi_\varepsilon(\phi_{\mathrm{p},\varepsilon}(x))$ for $x \in [-L_\varepsilon, x_0]$. By [26, Corollary 1] the estimates (2.16) yield

$$\|a_{\mathrm{p},\varepsilon}(x)\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad \|b_{\mathrm{p},\varepsilon}(x)\| \leq Ce^{\mu_0(x+x_0)}, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.19}$$

We also express the pulse solution $\phi_{\mathrm{h}}(x, u_0)$ to the fast reduced system (2.4) in Fenichel coordinates as $\tilde{\phi}_{\mathrm{h}}(x) = \Psi_0(\phi_{\mathrm{h}}(x, u_0))$ for $x \leq -x_0$. Observe that $\tilde{\phi}_{\mathrm{h}}(x)$ satisfies (2.15) for $\varepsilon = 0$ and lies in the unstable space $a = 0$ for $x \leq -x_0$. Consequently, we can write $\tilde{\phi}_{\mathrm{h}}(x) = (0, b_{\mathrm{h}}(x), c_0)$, where $c_0$ is a constant in

$U_F$ and $b_h(x)$ satisfies the equation $b_x = B(0,b,c_0,0)b$. Clearly, $b_h(x)$ converges exponentially to 0 as $x \to -\infty$. By estimate (2.18) and $C^1$-smoothness of $\Psi_\varepsilon$ in $\varepsilon$, it holds

$$\left\|\tilde{\phi}_{p,\varepsilon}(-x_0) - \tilde{\phi}_h(-x_0)\right\| \leq C\varepsilon. \tag{2.20}$$

Using estimates (2.17), (2.19) and (2.20) we obtain

$$\begin{aligned} \|c_{p,\varepsilon}(x) - c_0\| &\leq \int_x^{-x_0} \left(\varepsilon\|K(c_{p,\varepsilon}(y),\varepsilon)\| + \left\|H(\tilde{\phi}_{p,\varepsilon}(y),\varepsilon)(a_{p,\varepsilon}(y)\otimes b_{p,\varepsilon}(y))\right\|\right) dy + \|c_{p,\varepsilon}(-x_0) - c_0\| \\ &\leq C\varepsilon\Xi_\theta(\varepsilon), \end{aligned} \tag{2.21}$$

for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. The difference $g_\varepsilon(x) = b_{p,\varepsilon}(x) - b_h(x)$ satisfies an inhomogeneous equation of the form

$$g_x = A_\varepsilon(x)g + h_\varepsilon(x),$$

where $A_\varepsilon(x)g_\varepsilon(x) = B(\tilde{\phi}_h(x),0)g_\varepsilon(x) + (B(0,b_{p,\varepsilon}(x),c_0,0) - B(\tilde{\phi}_h(x),0))b_{p,\varepsilon}(x)$ and $h_\varepsilon(x) = (B(\tilde{\phi}_{p,\varepsilon}(x),\varepsilon) - B(0,b_{p,\varepsilon}(x),c_0,0))b_{p,\varepsilon}(x)$. Taking $x_0$ larger if necessary, estimates (2.16), (2.19) and (2.21) imply that $\mathrm{Re}(A_\varepsilon(x)) \leq -\mu_0$ and $\|h_\varepsilon(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon)$ for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. Therefore, we conclude using (2.20) that

$$\|b_{p,\varepsilon}(x) - b_h(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon), -x_0]. \tag{2.22}$$

Estimate (2.13) now follows from $C^1$-smoothness of $\Psi_\varepsilon^{-1}$ in $\varepsilon$ together with estimates (2.18), (2.19), (2.21) and (2.22).

We prove (2.14). By (2.17) and (2.19) we have

$$\left\|H(\tilde{\phi}_{p,\varepsilon}(x))(a_{p,\varepsilon}(x)\otimes b_{p,\varepsilon}(x))\right\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad x \in [-L_\varepsilon, -x_0].$$

Therefore, using Grönwall type estimates, there exists a solution $(0,0,c_{s,\varepsilon}(x))$ to (2.15) on the invariant manifold $a = b = 0$, which is $O(e^{-\mu_0 L_\varepsilon})$-close to $c_{p,\varepsilon}(x)$ for $x \in [-L_\varepsilon, -x_0]$. The $c$-coordinate $c_{s,\varepsilon}(x)$ solves $c_x = \varepsilon K(c,\varepsilon)$ and is to leading order described by a solution $c_{s,0}(\check{x})$ to $c_{\check{x}} = K(c,0)$. This results in the estimate

$$\|c_{p,\varepsilon}(x) - c_{s,0}(\varepsilon x)\| \leq C\varepsilon, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.23}$$

Estimates (2.21) and (2.23) imply $c_{s,0}(0) = c_0$. On the other hand, we must have $\Psi_0((\psi_s(2\ell_0),0)) = \lim_{x\to-\infty}\tilde{\phi}_h(x) = (0,0,c_0)$. Since system $c_{\check{x}} = K(c,0)$ corresponds to the slow reduced system (2.6), we have $\Psi_0^{-1}((0,0,c_{s,0}(\check{x}))) = (\psi_s(\check{x}+2\ell_0),0)$ for $\varepsilon^{-1}\check{x} \in [-L_\varepsilon, 0]$. Hence, by $C^1$-smoothness of $\Psi_\varepsilon^{-1}$ in $\varepsilon$, $R$-reversibility of $\phi_{p,\varepsilon}(x)$, estimates (2.19) and (2.23) and the inequality $\theta \geq \mu_0^{-1}$, we conclude estimate (2.14) holds true.

Finally, we prove the first assertion. On the one hand, we have $p_s(\ell_0) = 0$ and $p_s'(\ell_0) \neq 0$ by **(E2)**. On the other hand, it holds $\|p_s(\varepsilon L_\varepsilon)\| \leq C\varepsilon$ by (2.14). Thus, an application of the inverse function theorem and the mean value theorem yields $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$. □

# 3 Main results

We assume that **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Theorem 2.2 provides a reversibly symmetric, $2L_\varepsilon$-periodic pulse solution $\phi_{p,\varepsilon}(x)$ to (2.3). This yields a stationary, periodic pulse solution $\hat{\phi}_{p,\varepsilon}(x) = (u_{p,\varepsilon}(x), v_{p,\varepsilon}(x))$ to system (2.2). We denote by $\check{\phi}_{p,\varepsilon}(\check{x})$ the corresponding solution to the rescaled system (2.1). The stability of $\check{\phi}_{p,\varepsilon}$ is determined by linearizing (2.1) about $\check{\phi}_{p,\varepsilon}$: we obtain the periodic differential operator $\mathcal{L}_\varepsilon$ on $C_{ub}(\mathbb{R},\mathbb{R}^{m+n})$ with domain $C^2_{ub}(\mathbb{R},\mathbb{R}^{m+n})$ given by

$$\mathcal{L}_\varepsilon\psi = D_\varepsilon\psi_{\check{x}\check{x}} - \mathcal{B}_\varepsilon\psi,$$

with

$$D_\varepsilon := \begin{pmatrix} D_1 & 0 \\ 0 & \varepsilon^2 D_2 \end{pmatrix},$$

and

$$\mathcal{B}_\varepsilon(\check{x}) := \begin{pmatrix} \partial_u H_1(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) + \varepsilon^{-1}\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}) & \partial_v H_1(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) + \varepsilon^{-1}\partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}) \\ \partial_u G(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) & \partial_v G(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) \end{pmatrix},$$

where we suppress the $\check{x}$-dependence of $\check{\phi}_{\mathrm{p},\varepsilon}$. Here, $C^k_{\mathrm{ub}}(\mathbb{R},\mathbb{R}^{m+n})$ denotes the space of $k$ times continuously differentiable functions with derivatives up to order $k$, which are bounded and uniformly continuous. Note that $\mathcal{L}_\varepsilon$ is closed, densely defined and sectorial [31]. Moreover, by Theorem 2.2, $\mathcal{L}_\varepsilon$ is a $2\ell_\varepsilon$-periodic differential operator, where $\ell_\varepsilon := \varepsilon L_\varepsilon \to \ell_0$ as $\varepsilon \to 0$ with $\ell_0 > 0$ defined in **(E2)**. Therefore, Floquet-Bloch decomposition [19] characterizes the spectrum of $\mathcal{L}_\varepsilon$. Consider the family of closed and densely defined operators $\mathcal{L}_{\nu,\varepsilon}$ on $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ given by

$$\mathcal{L}_{\nu,\varepsilon}\psi = D_\varepsilon\left(\partial_{\check{x}} - \frac{i\nu}{2\ell_\varepsilon}\right)^2\psi - \mathcal{B}_\varepsilon\psi, \quad \nu \in [-\pi,\pi],$$

where $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ is the space of $L^2$-integrable $2\ell_0$-periodic functions. Since $\mathcal{L}_{\nu,\varepsilon}$ has compact resolvent, its spectrum is discrete and consists entirely of eigenvalues. The spectrum of $\mathcal{L}_\varepsilon$ is given by the union

$$\sigma(\mathcal{L}_\varepsilon) = \bigcup_{\nu\in[-\pi,\pi]} \sigma(\mathcal{L}_{\nu,\varepsilon}). \tag{3.1}$$

The spectral decomposition (3.1) gives rise to the following definition.

**Definition 3.1.** Let $\nu \in [-\pi,\pi]$ and $\gamma = e^{i\nu} \in S^1$. A point $\lambda \in \sigma(\mathcal{L}_{\nu,\varepsilon})$ is called a $\gamma$-*eigenvalue* of $\mathcal{L}_\varepsilon$. The algebraic multiplicity of $\lambda$ as an eigenvalue of the operator $\mathcal{L}_{\nu,\varepsilon}$ is the *algebraic $\gamma$-multiplicity* of $\lambda$ as a $\gamma$-eigenvalue of $\mathcal{L}_\varepsilon$.

Thus, the spectrum of $\mathcal{L}_\varepsilon$ is a union of curves parameterized over the unit circle $S^1$. Due to translational invariance, it holds $\check{\phi}'_{\mathrm{p},\varepsilon} \in \ker(\mathcal{L}_{0,\varepsilon})$. Therefore, one of the spectral curves is attached to the origin.

When the spectrum of $\mathcal{L}_\varepsilon$ is confined to the left half-plane and bounded away from the imaginary axis, except for a quadratic tangency at the origin, it is known [27, 45, 46] that the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinear diffusively stable as solution to (2.1). Verifying such spectral conditions is in general very hard, especially for multi-component systems. However, as mentioned in the introduction, the presence of the small parameter $\varepsilon$ in (2.1) provides a mechanism to reduce complexity. On the one hand, the Evans-function analysis in [8] yields asymptotic control over the spectrum, i.e. in the limit $\varepsilon \to 0$ the spectrum of $\mathcal{L}_\varepsilon$ is given by the roots of an explicit reduced Evans function, which is defined in terms of simpler, lower-dimensional eigenvalue problems. On the other hand, the critical spectral curve attached to the origin shrinks to the origin in the limit $\varepsilon \to 0$. Thus, controlling the spectrum in the limit $\varepsilon \to 0$ is insufficient to establish nonlinear stability. Therefore, we complement the analysis in [8] in this paper by providing an expansion of this critical spectral curve. This leads to sufficient control over the spectrum to establish nonlinear stability or instability depending on the system parameters.

This section is structured as follows. First, we provide conditions on the spectrum of $\mathcal{L}_\varepsilon$ for nonlinear stability. Subsequently, we outline the results of [8] yielding control over the spectrum in the limit $\varepsilon \to 0$. Next, we state the main result of this paper: a rigorous expansion of the critical spectral curve touching the origin. Together with the results in [8], this leads to explicit criteria for stability and instability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ in terms of simpler, lower-dimensional (eigenvalue) problems. We further simplify these criteria in the case $n = 1$ or $m = 1$.

## 3.1 Nonlinear stability and instability by linear approximation

In this subsection we collect nonlinear (in)stability results from the literature. More precisely, we present conditions on the spectrum of the linearization $\mathcal{L}_\varepsilon$ of (2.1) about $\check{\phi}_{\mathrm{p},\varepsilon}$ yielding some form of nonlinear stability or instability.

As mentioned before, 0 is always a 1-eigenvalue of $\mathcal{L}_\varepsilon$, since the derivative $\check{\phi}'_{\mathrm{p},\varepsilon}$ is contained in $\ker(\mathcal{L}_{0,\varepsilon})$. If we assume that 0 has algebraic 1-multiplicity 1, then there exists by the implicit function theorem a spectral curve $\lambda_\varepsilon \colon U_\varepsilon \to \mathbb{C}$, where $U_\varepsilon \subset [-\pi, \pi]$ is a neighborhood of 0, such that $\lambda_\varepsilon(0) = 0$ and $\lambda_\varepsilon(\nu)$ is a $e^{i\nu}$-eigenvalue for $\nu \in U_\varepsilon$. If this critical spectral curve touches the origin in a quadratic tangency and the rest of the spectrum is confined to the left half-plane, bounded away from the imaginary axis, then one obtains some form of nonlinear stability. This leads to the following definition.

**Definition 3.2.** The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1) is *spectrally stable* if 0 is a simple eigenvalue of $\mathcal{L}_{0,\varepsilon}$ and there exists $\varsigma > 0$, possibly dependent on $\varepsilon$, such that

$$\mathrm{Re}(\lambda_\varepsilon(\nu)) \leq -\varsigma \nu^2, \quad \nu \in U_\varepsilon,$$
$$\sigma(\mathcal{L}_\varepsilon) \setminus \lambda_\varepsilon[U_\varepsilon] \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\varsigma\}.$$

Spectral stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ implies nonlinear diffusive stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ with respect to localized perturbations. In addition, an initial displacement of the periodic pulse can be tracked for large times.

**Theorem 3.3.** *[45, Theorem 1] Suppose $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable. Take $b \in (0, \frac{1}{2})$. There are $\delta, C > 0$, possibly dependent on $\varepsilon$, such that the following holds. The solution $\check{\phi}(x,t)$ to (2.1) with initial condition*

$$\check{\phi}(\check{x}, 0) = \check{\phi}_{\mathrm{p},\varepsilon}(\check{x} + \theta_0(\check{x})) + v_0(\check{x}),$$

*with $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ and $\theta_0 \in H^3(\mathbb{R}, \mathbb{R})$ satisfying $\|\theta_0 \rho\|_{H^3}, \|v_0 \rho\|_{H^2} \leq \delta$ with $\rho(\check{x}) = (1 + \check{x}^2)^{3/2}$, exists for all times $t \geq 0$ and can be written as*

$$\check{\phi}(\check{x}, t) = \check{\phi}_{\mathrm{p},\varepsilon}(\check{x} + \theta(\check{x}, t)) + v(\check{x}, t), \quad t > 0,$$

*where $\theta \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ and $v \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}^{m+n}$. There exists a constant $\theta_{\lim} \in \mathbb{R}$ such that*

$$\sup_{\check{x} \in \mathbb{R}} \left[ |\theta(\check{x}, t) - \theta_{\lim} G(\check{x}, t)| + \|v(\check{x}, t)\| \right] \leq C(1+t)^{-1+b}, \quad t > 0,$$

*where G is the Gaussian*

$$G(\check{x}, t) = \frac{1}{\sqrt{4\alpha\pi(1+t)}} e^{-\check{x}^2/(4\alpha(1+t))},$$

*with $\alpha := -\lambda''_\varepsilon(0)$. In particular, we have*

$$\sup_{\check{x} \in \mathbb{R}} \left\| \check{\phi}(\check{x}, t) - \check{\phi}_{\mathrm{p},\varepsilon}\left(\check{x} + \theta_{\lim} G(\check{x}, t)\right) \right\| \leq C(1+t)^{-1+b}, \quad t > 0.$$

The above result is to be compared with [46, Theorem 1.1]. Here, the class of allowed perturbations is larger, i.e. one requires $v_0 \tilde{\rho} \in H^{1/2+b}(\mathbb{R}, \mathbb{R}^{m+n})$ with $\tilde{\rho}(\check{x}) = 1 + \check{x}^2$. However, in [46] one obtains a weaker decay bound of the form $\sup_{\check{x} \in \mathbb{R}} \|v(\check{x}, t)\| \leq C(1+t)^{-1/2}$. Moreover, in [27] pointwise estimates are obtained with respect to perturbations $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ satisfying $\|v_0(\check{x})\| \leq E_0 e^{-\check{x}^2/M}$ or $\|v_0(\check{x})\| \leq E_0(1 + |\check{x}|)^{-r}$ for some $M > 1, r > 2$ and $E_0 > 0$. The decay rates obtained in [27] are comparable to those in Theorem 3.3, yet they are more specific, since they depend pointwise on $\check{x}$. Finally, we emphasize that both in [27] and [46] one does not consider an initial displacement in time in contrast to Theorem 3.3.

Spectrum of $\mathcal{L}_\varepsilon$ in the *right* half-plane yields nonlinear instability of the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ against localized and non-localized perturbations.

**Definition 3.4.** The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1) is *spectrally unstable* if there exists $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda) > 0$.

**Theorem 3.5.** *[32, Section 4] Let $X = H^2(\mathbb{R}, \mathbb{R}^{m+n})$ or $X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$. Suppose the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (2.1) *is spectrally unstable. Then, there exists $\delta > 0$ and a sequence of solutions $\check{\phi}_n(\check{x}, t), n \in \mathbb{Z}_{>0}$ to* (2.1) *satisfying $\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon} \in X$ such that*

$$\left\|\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X \to 0 \text{ as } n \to \infty,$$

*but for all $n \in \mathbb{Z}_{>0}$ there exists $t_n > 0$ such that*

$$\begin{aligned} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X &\geq \delta, \text{ in the case } X = H^2(\mathbb{R}, \mathbb{R}^{m+n}), \\ \inf_{\theta \in \mathbb{R}} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)\right\|_X &\geq \delta, \text{ in the case } X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n}). \end{aligned}$$

We emphasize that in the case of non-localized perturbations, it is important to measure the distance from the perturbation to the family of all translates of the solution rather than to the solution itself. Indeed, any translate $\check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)$ corresponds to a non-localized perturbation. Yet, such a translate is a solution to (2.1) itself. Thus, $\check{\phi}_{\mathrm{p},\varepsilon}$ is never stable against translation of the profile. We stress that the $\theta$-terms in Theorem 3.3 account for translation of the profile.

### 3.2 Controlling the spectrum in the limit $\varepsilon \to 0$ through the Evans function

We outline the results of the companion paper [8] in two steps. First, we introduce the Evans function as a tool to locate the spectrum of the linearization $\mathcal{L}_\varepsilon$. Second, we establish an explicit *reduced Evans function*, whose roots determine the spectrum $\sigma(\mathcal{L}_\varepsilon)$ in the limit $\varepsilon \to 0$.

Recall that a point $\lambda \in \mathbb{C}$ is in the spectrum of $\mathcal{L}_\varepsilon$ if and only if there exists $\psi \in C^2_{ub}(\mathbb{R}, \mathbb{C}^{m+n})$ such that $\mathcal{L}_\varepsilon \psi = \lambda \psi$. The latter equation can be rewritten as an ODE in the 'small' spatial scale $x = \varepsilon^{-1}\check{x}$ as follows

$$\varphi_x = \mathcal{A}_\varepsilon(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{3.2}$$

with coefficient matrix

$$\mathcal{A}_\varepsilon(x, \lambda) := \begin{pmatrix} \mathcal{A}_{11,\varepsilon}(x, \lambda) & \mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix},$$

where the blocks are given by

$$\begin{aligned} \mathcal{A}_{11,\varepsilon}(x, \lambda) &:= \begin{pmatrix} 0 & \varepsilon D_1^{-1} \\ \varepsilon\left(\partial_u H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) + \lambda\right) + \partial_u H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\ \mathcal{A}_{12,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \varepsilon \partial_v H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) + \partial_v H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\ \mathcal{A}_{21,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) & 0 \end{pmatrix}, \\ \mathcal{A}_{22,\varepsilon}(x, \lambda) &:= \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) + \lambda & 0 \end{pmatrix}, \end{aligned}$$

and $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$ is the $2L_\varepsilon$-periodic pulse solution to (2.2). We will refer to (3.2) as the *full eigenvalue problem*. By Floquet Theory bounded solutions to (3.2) must satisfy $\varphi(-L_\varepsilon) = \gamma\varphi(L_\varepsilon)$ for some $\gamma \in S^1$. This fact leads to the definition of the Evans function.

**Definition 3.6.** Denote by $\mathcal{T}_\varepsilon(x, z, \lambda)$ the evolution operator of system (3.2). The *Evans function* $\mathcal{E}_\varepsilon\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is given by

$$\mathcal{E}_\varepsilon(\lambda, \gamma) := \det(\mathcal{T}_\varepsilon(0, -L_\varepsilon, \lambda) - \gamma\mathcal{T}_\varepsilon(0, L_\varepsilon, \lambda)).$$

**Proposition 3.7.** *[8, Section 3.1] The Evans function has the following properties:*

1. *The Evans function is analytic in both $\lambda$ and $\gamma$;*
2. *We have $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ if and only if there exists $\gamma \in S^1$ such that $\mathcal{E}_\varepsilon(\lambda, \gamma) = 0$. In that case, $\lambda$ is a $\gamma$-eigenvalue and its algebraic $\gamma$-multiplicity is equal to the multiplicity of $\lambda$ as a root of $\mathcal{E}_\varepsilon(\cdot, \gamma)$;*
3. *It holds $\overline{\mathcal{E}_\varepsilon(\lambda, \gamma)} = \mathcal{E}_\varepsilon(\overline{\lambda}, \overline{\gamma})$ for $\lambda, \gamma \in \mathbb{C}$. Thus, the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is invariant under complex conjugation;*
4. *We have $\mathcal{E}_\varepsilon(\lambda, \gamma) = \mathcal{E}_\varepsilon(\lambda, \overline{\gamma})\gamma^{2(m+n)}$ for $\lambda \in \mathbb{C}$ and $\gamma \in S^1$. Thus, $\lambda$ is a $\gamma$-eigenvalue if and only if it is a $\overline{\gamma}$-eigenvalue.*

Proposition 3.7 yields that the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is an (at most countable) union of curves, each of which is covered twice by the unit circle $S^1$. The endpoints of the curves are $\pm1$-eigenvalues. Proposition 3.7 and the implicit function theorem yield the following reformulation of the concept 'spectral stability' introduced in §3.1.

**Corollary 3.8.** *The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (2.1) *is spectrally stable if and only if*

*i.* $\mathcal{E}_\varepsilon(\lambda, \gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

*ii.* $\mathcal{E}_\varepsilon(0, \gamma) \neq 0$ *for all* $\gamma \in S^1 \setminus \{1\}$*;*

*iii.* $\partial_\lambda\mathcal{E}_\varepsilon(0, 1)\partial_{\gamma\gamma}\mathcal{E}_\varepsilon(0, 1) < 0$.

For the introduction of the reduced Evans function we need the following technical lemma.

**Lemma 3.9.** *[8, Lemma 3.7] Let $\mathcal{K} \subset \mathbb{C}^m$ be an open and bounded set containing the orbit segment $\{u_s(\check{x}) : \check{x} \in [0, 2\ell_0]\}$ such that $\overline{\mathcal{K}} \subset U$ – see* ***(S1)*** *and* ***(E2)****. Moreover, denote*

$$C_\Lambda := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \Lambda\}, \qquad \Lambda \in \mathbb{R}.$$

*There exists $\Lambda_0 > 0$ such that for $\Lambda \in (-\Lambda_0, 0)$ the spectrum of the matrix*

$$A(u, \lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, 0, 0) + \lambda & 0 \end{pmatrix}, \tag{3.3}$$

*is bounded away from the imaginary axis on $\overline{\mathcal{K}} \times C_\Lambda$ by some constant $\mu_r > 0$.*

Lemma 3.9 shows that the spectrum of the coefficient matrix of (3.2) admits a consistent splitting as long as the $v$-components of the periodic pulse solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ are sufficiently small. This splitting is essential for the approximation in [8] of the roots of the Evans function by the ones of the reduced Evans function, but also for the spectral analysis in this paper – see §5.

Now take $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 3.9. The reduced Evans function $\mathcal{E}_0\colon C_\Lambda \times \mathbb{C} \to \mathbb{C}$ is defined as the product

$$\mathcal{E}_0(\lambda, \gamma) = (-\gamma)^n\mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda, \gamma).$$

Here, the analytic map $\mathcal{E}_{f,0}\colon C_\Lambda \to \mathbb{C}$ is called the *fast Evans function*. It is associated with the *homogeneous fast eigenvalue problem*

$$\varphi_x = \mathcal{A}_{22,0}(x, u_0, \lambda)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.4}$$

with

$$\mathcal{A}_{22,0}(x, u, \lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, v_\mathrm{h}(x, u), 0) + \lambda & 0 \end{pmatrix}, \quad u \in U_\mathrm{h}.$$

Recall that $U_\mathrm{h}, v_\mathrm{h}(x, u), u_0 = u_\mathrm{s}(0)$ and $u_\mathrm{s}(\check{x})$ are defined in **(E1)** and **(E2)**. Note that equation (3.4) is equivalent to $\mathcal{L}_f\varphi = \lambda\varphi$, where $\mathcal{L}_f\colon L^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ is the closed, densely defined and sectorial operator given by

$$\mathcal{L}_f v = D_2 v_{xx} - \partial_v G(u_0, v_\mathrm{h}(\cdot, u_0), 0)v. \tag{3.5}$$

We establish the existence of the fast Evans function.

**Proposition 3.10.** *[8, Section 6.3] There exists an analytic map $\mathcal{E}_{f,0}\colon C_\Lambda \to \mathbb{C}$, which has a zero if and only if* (3.4) *admits a non-trivial, exponentially localized solution. In particular, the multiplicity of a root $\lambda \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of $\lambda$ as an eigenvalue of the sectorial operator $\mathcal{L}_f$, which is Fredholm of index* 0.

**Remark 3.11.** Note that the homogeneous fast eigenvalue problem (3.4) at $\lambda = 0$ equals the variational equation

$$\varphi_x = \mathcal{A}_f(x)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.6}$$

about the homoclinic solution $\psi_\mathrm{h}(x, u_0)$ to (2.5) at $u = u_0$ with

$$\mathcal{A}_f(x) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0, v_\mathrm{h}(x, u_0), 0) & 0 \end{pmatrix}.$$

We will refer to (3.6) as the *fast variational equation*. Clearly, the derivative of the homoclinic solution $\partial_x\psi_\mathrm{h}(x, u_0)$ is a non-trivial, exponentially localized solution to (3.4) at $\lambda = 0$ by **(S2)** and **(E1)**. Thus, it holds $\mathcal{E}_{f,0}(0) = 0$ by Proposition 3.10. Now assume 0 is a simple root of $\mathcal{E}_{f,0}$. Then, since $\mathcal{L}_f$ is Fredholm of index 0, both $\ker(\mathcal{L}_f)$ and $\ker(\mathcal{L}_f^*)$ are one-dimensional by Proposition 3.10. Thus, the associated adjoint problem

$$\varphi_x = -\mathcal{A}_f(x)^*\varphi, \quad \varphi \in \mathbb{R}^{2n}. \tag{3.7}$$

admits a non-trivial, bounded solution $\psi_\mathrm{ad}(x)$, which is unique up to scalar multiples. Since the asymptotic matrix $A(u_0, 0) = \lim_{x\to\pm\infty} \mathcal{A}_f(x)$ is hyperbolic by Lemma 3.9, $\psi_\mathrm{ad}(x)$ must be exponentially decaying as $x \to \pm\infty$. ■

The slow Evans function $\mathcal{E}_{s,0}\colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is determined by two eigenvalue problems. The first is the *inhomogeneous fast eigenvalue problem*

$$\partial_x \mathcal{X} = \mathcal{A}_{22,0}(x, u, \lambda)\mathcal{X} + \mathcal{A}_{21,0}(x, u), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}), \tag{3.8}$$

with

$$\mathcal{A}_{21,0}(x, u) := \begin{pmatrix} 0 & 0 \\ \partial_u G(u, v_\mathrm{h}(x, u), 0) & 0 \end{pmatrix}, \quad u \in U_\mathrm{h}.$$

The second is the *slow eigenvalue problem*

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= (\partial_u H_1(u_s(\check{x}), 0, 0) + \lambda)\, u, \end{aligned} \qquad (u, p) \in \mathbb{C}^{2m}. \tag{3.9}$$

The *slow Evans function* $\mathcal{E}_{s,0}\colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is defined by

$$\mathcal{E}_{s,0}(\lambda, \gamma) = \det\left(\Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda) - \gamma I\right),$$

where $\ell_0 > 0$ is as in **(E2)**, $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ is the evolution operator of the slow eigenvalue problem (3.9) and $\Upsilon(u, \lambda)$ is given by

$$\begin{aligned} \Upsilon(u, \lambda) &= \begin{pmatrix} I & 0 \\ \mathcal{G}(u, \lambda) & I \end{pmatrix}, \\ \mathcal{G}(u, \lambda) &= \int_{-\infty}^{\infty} [\partial_u H_2(u, v_h(x, u)) + \partial_v H_2(u, v_h(x, u))\mathcal{V}_{in}(x, u, \lambda)]\, dx, \end{aligned} \qquad u \in U_h,$$

where $\mathcal{V}_{in}(x, u, \lambda)$ denotes the upper-left $(n \times m)$-block of the unique matrix solution $\mathcal{X}_{in}(x, u, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8). We collect some properties of the slow Evans function.

**Proposition 3.12.** *[8, Section 6.3] The slow Evans function $\mathcal{E}_{s,0}\colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is well-defined and enjoys the following properties:*

1. *$\mathcal{E}_{s,0}$ is analytic on its domain;*
2. *$\mathcal{E}_{s,0}(\cdot, \gamma)$ is meromorphic on $C_\Lambda$ for each $\gamma \in \mathbb{C}$ in such a way that the reduced Evans function $\mathcal{E}_0$ is analytic on its domain;*
3. *$\mathcal{E}_{s,0}(\lambda, \cdot)$ is a polynomial of degree $2m$ and it holds $\mathcal{E}_{s,0}(\lambda, \gamma) = \gamma^{2m}\mathcal{E}_{s,0}(\lambda, \overline{\gamma})$ for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ and $\gamma \in S^1$;*
4. *The set of roots,*

   $$\bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_{s,0}(\lambda, \gamma) = 0\},$$

   *is bounded;*
5. *If $0$ is a simple zero of $\mathcal{E}_{f,0}$, then $\mathcal{E}_{s,0}(\lambda, \gamma)$ has a removable singularity at $\lambda = 0$ for each $\gamma \in S^1$.*

Finally, we present the main result of [8]. It concerns the approximation of the zeros of $\mathcal{E}_\varepsilon$ by the ones of $\mathcal{E}_0$.

**Theorem 3.13.** *[8, Section 3.2] Let $S \subset S^1$ be a closed subset and take a simple closed curve $\Gamma$ in $C_\Lambda \setminus \mathcal{N}_S$, where*

$$\mathcal{N}_S := \bigcup_{\gamma \in S} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda, \gamma) = 0\}.$$

*Then, for $\varepsilon > 0$ sufficiently small, the number of roots (including multiplicity) of $\mathcal{E}_0(\cdot, \gamma)$ and $\mathcal{E}_\varepsilon(\cdot, \gamma)$ interior to $\Gamma$ coincides for any $\gamma \in S^1$.*

Theorem 3.13 provides the (critical) spectrum of the operator $\mathcal{L}_\varepsilon$ in the limit $\varepsilon \to 0$ by taking $S = S^1$. On the other hand, by taking $S = \{e^{i\nu}\}$ for some $\nu \in \mathbb{R}$, one establishes the convergence of the spectrum $\sigma(\mathcal{L}_{\nu,\varepsilon})$ to the discrete set $\{\lambda \in \mathbb{C} : \mathcal{E}_0(\lambda, e^{i\nu}) = 0\}$.

### 3.3 Expansion of the critical spectral curve

Theorem 3.13 provides control over the spectrum of the operator $\mathcal{L}_\varepsilon$ in the limit $\varepsilon \to 0$. However, as mentioned in the introduction in §1, this is insufficient to establish spectral stability, since the critical spectral curve attached to the origin shrinks to the origin as $\varepsilon \to 0$. Hence, we cannot determine with Theorem 3.13 whether the critical curve lies in the left half-plane and touches the origin in a quadratic tangency – see Definition 3.2. This brings us to the main result of this paper that provides leading-order control over the critical spectral curve.

**Theorem 3.14.** *Suppose* $0$ *is a simple zero of* $\mathcal{E}_{f,0}$. *Let* $\delta > 0$ *and denote*

$$\mathcal{N}_\diamond := \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta := \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta). \tag{3.10}$$

*Then, for* $\varepsilon > 0$ *sufficiently small, there exists for any* $\nu \in \mathcal{S}_\delta$ *a unique (real-valued) root* $\lambda_\varepsilon(\nu)$ *of* $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ *converging to* $0$ *as* $\varepsilon \to 0$. *The function* $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ *is analytic, even,* $2\pi$*-periodic and satisfies* $\lambda_\varepsilon(0) = 0$ *if* $0 \in \mathcal{S}_\delta$. *Moreover,* $\lambda_\varepsilon(\nu)$ *is approximated as*

$$\left|\lambda_\varepsilon(\nu) - \varepsilon^2 \lambda_0(\nu)\right| \leq C\varepsilon^3 |\log(\varepsilon)|^5, \tag{3.11}$$

*where* $C > 0$ *is a constant independent of* $\varepsilon$ *and* $\nu$ *and the analytic function* $\lambda_0 \colon \mathcal{N}_\diamond \to \mathbb{R}$ *is given by*

$$\lambda_0(\nu) := \frac{\int_{-\infty}^{\infty} \langle \partial_u G(u_0, v_{\rm h}(x, u_0), 0)^* \psi_{{\rm ad},2}(x) x, B(\nu)\rangle \, dx}{\int_{-\infty}^{\infty} \langle \psi_{{\rm ad},2}(x), \partial_x v_{\rm h}(x, u_0)\rangle \, dx}, \tag{3.12}$$

*with* $\psi_{\rm ad}(x) = (\psi_{{\rm ad},1}(x), \psi_{{\rm ad},2}(x))$ *a non-trivial, exponentially localized solution to* (3.7) *and*

$$\begin{aligned}
B(\nu) &:= D_1^{-1} \begin{pmatrix} 0 & I \end{pmatrix} \mathcal{B}(\nu), \\
\mathcal{B}(\nu) &:= \Upsilon_0^{-1} \begin{pmatrix} D_1^{-1} \mathcal{J}(u_0) \\ H_1(u_0, 0, 0) \end{pmatrix} - \left(I - e^{-i\nu} \Upsilon_0 \Phi_{\rm s}(2\ell_0, 0) \Upsilon_0\right)^{-1} \begin{pmatrix} 2D_1^{-1} \mathcal{J}(u_0) \\ 0 \end{pmatrix}, \\
\Upsilon_0 &:= \begin{pmatrix} I & 0 \\ \partial_u \mathcal{J}(u_0) & I \end{pmatrix},
\end{aligned} \tag{3.13}$$

*where* $\mathcal{J} \colon U_{\rm h} \to \mathbb{R}^m$ *is defined in* (2.7) *and* $\Phi_{\rm s}(\check{x}, \check{y})$ *is the evolution operator of the slow variational equation* (2.9).

**Remark 3.15.** Since 0 is a simple root of $\mathcal{E}_{f,0}$, Proposition 3.10 implies that 0 is an (algebraically) simple eigenvalue of the operator $\mathcal{L}_f$, which is defined in (3.5). Therefore, the associated generalized eigenvalue problem has no solution and the solvability condition

$$\int_{-\infty}^{\infty} \langle \psi_{{\rm ad},2}(x), \partial_x v_{\rm h}(x, u_0)\rangle \, dx \neq 0,$$

is satisfied – see Remark 3.11 and Proposition 5.1. Thus, the denominator of $\lambda_0(\nu)$ in (3.12) does not vanish. ■

**Remark 3.16.** If we have $\mathcal{E}_{s,0}(0, e^{i\nu_\diamond}) = 0$ for some $\nu_\diamond \in \mathbb{R}$, then the approximation of $\lambda_\varepsilon(\nu)$ in Theorem 3.14 fails for $\nu$ in a neighborhood of $\nu_\diamond$. Since it holds $\det\left(I - e^{-i\nu} \Upsilon_0 \Phi_{\rm s}(2\ell_0, 0) \Upsilon_0\right) = \mathcal{E}_{s,0}(0, e^{-i\nu}) = e^{2im\nu} \mathcal{E}_{s,0}(0, e^{i\nu}) = 0$ by Proposition 3.12, we observe that $\lambda_0$ has a pole at $\nu_\diamond$. ■

The critical spectral curve $\lambda_\varepsilon(\nu)$ arises as the solution curve to the equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ about $(\lambda, \nu) = (0, 0)$. The equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ is defined in terms of the $2(m+n)$-dimensional full eigenvalue problem (3.2). The leading-order approximation $\lambda_0(\nu)$ of the solution curve $\lambda_\varepsilon(\nu)$, established in Theorem 3.14, is defined in terms of the $\varepsilon$-independent, $2m$-dimensional slow variational equation (2.9) and

$2n$-dimensional fast variational equation (3.6). Therefore, Theorem 3.14 yields a reduction of complexity in the local analysis of the full eigenvalue problem (3.2) about $\lambda = 0$. Combining this with Theorem 3.13 leads to a set of spectral stability conditions in terms of simpler, lower-dimensional systems, which we will present in the next subsection.

The proof of Theorem 3.14 is provided in §5.

**Remark 3.17.** Suppose the conditions in Theorem 3.14 are met. Observe that the derivative $\psi_{\mathrm{s}}'$ of the solution $\psi_{\mathrm{s}}$ to (2.6) is a solution to the slow variational equation (2.9). By assumption **(E2)** $\psi_{\mathrm{s}}(\check{x})$ intersects the touch-down manifold $\mathcal{T}_+$ at $\check{x} = 0$ in the point $(u_0, \mathcal{J}(u_0))$. Therefore, we have $\psi_{\mathrm{s}}'(0) = (D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$ and by reversible symmetry: $\psi_{\mathrm{s}}'(2\ell_0) = R_s\psi_{\mathrm{s}}'(0) = (-D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$. Thus, we deduce

$$\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} = \begin{pmatrix} -D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} + (\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0) - I)\begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix},$$

where $\mathfrak{a} := \partial_u\mathcal{J}(u_0)D_1^{-1}\mathcal{J}(u_0) - H_1(u_0, 0, 0) \in \mathbb{R}^m$. Rewriting the latter equation gives

$$\begin{aligned} &\left(1 + e^{-i\nu}\right)\left(I - e^{-i\nu}\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} \\ &\qquad = e^{-i\nu}\left(I - e^{-i\nu}\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right)^{-1}(\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0) - I)\begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix} + \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix}, \end{aligned}$$

for $\nu \in [0, \pi]$. Hence, we obtain the following expression for the quantity $B(\nu)$ in Theorem 3.14

$$B(\nu) = D_1^{-1}\left[\frac{\begin{pmatrix} 0 & I \end{pmatrix}}{1 + e^{-i\nu}}\left(I - e^{-i\nu}\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right)^{-1}(I - \Upsilon_0\Phi_{\mathrm{s}}(2\ell_0, 0))\begin{pmatrix} 0 \\ 2I \end{pmatrix} - I\right]\mathfrak{a},$$

for $\nu \in [0, \pi)$. So, $\mathfrak{a} = 0$ implies $\lambda_0(\nu) = 0$ for any $\nu \in [0, \pi)$ by Theorem 3.14. Therefore, $\mathfrak{a}$ passing through zero, suggests a transition of the critical spectral curve through the imaginary axis. This coincides with a loss of transversality: condition (2.10) in assumption **(E2)** fails if $\mathfrak{a} = 0$. For the case $m = n = 1$, we show in §4 that the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ destabilizes through a spatial period doubling bifurcation or sideband instability as $\mathfrak{a}$ passes through zero. ■

## 3.4 Explicit criteria for spectral stability and instability

Using Theorems 3.13 and 3.14, we obtain explicit conditions in terms of simpler, lower-dimensional problems yielding spectral stability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1). First, we employ Theorem 3.13 to isolate the most critical part of the spectrum – i.e. the curve $\lambda_\varepsilon(\nu)$ attached to the origin – and to bound the rest of the spectrum away from the right half-plane. Subsequently, we use the expansion in Theorem 3.14 to control the critical curve attached to the origin. Thus, we obtain the following result.

**Corollary 3.18.** *Suppose the following conditions are met:*

*i.* $0$ *is a simple zero of* $\mathcal{E}_{f,0}$*;*

*ii.* $\mathcal{E}_{s,0}(0, \gamma) \neq 0$ *for each* $\gamma \in S^1$*;*

*iii.* $\mathcal{E}_0(\lambda, \gamma) \neq 0$ *for each* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

*iv.* $\lambda_0''(0) < 0$, $\lambda_0(\pi) < 0$ *and* $\lambda_0'(\nu) \neq 0$ *for each* $\nu \in (0, \pi)$*, where* $\lambda_0 \colon \mathbb{R} \to \mathbb{R}$ *is defined by* (3.12).

*Then, provided* $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (2.1) *is spectrally stable.*

**Proof.** Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 3.9. The set

$$\mathcal{N}_0 := \bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0\},$$

is bounded by Propositions 3.10 and 3.12. So, there exists $\sigma_0 > 0$ such that, if $\mathcal{E}_0(\lambda,\gamma) = 0$ is satisfied for some $\gamma \in S^1$ and $\lambda \in C_\Lambda \setminus \{0\}$, then we must have $\mathrm{Re}(\lambda) < -\sigma_0$. In addition, 0 is a simple zero of $\mathcal{E}_0(\cdot,\gamma)$ for each $\gamma \in S^1$ by Proposition 3.12. Let $\delta \in (0,\sigma_0)$. Theorem 3.13 implies that, provided $\varepsilon > 0$ is sufficiently small, $\mathcal{E}_\varepsilon(\cdot,\gamma)$ has no roots in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} \setminus B(0,\delta)$ for each $\gamma \in S^1$. Moreover, for each $\gamma \in S^1$ there is a unique simple zero of $\mathcal{E}_\varepsilon(\cdot,\gamma)$ in $B(0,\delta)$. Combining this with Proposition 3.7 and Theorem 3.14 proves

$$\sigma(\mathcal{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} = \lambda_\varepsilon[\mathbb{R}].$$

Since both $\lambda_\varepsilon$ and $\lambda_0$ are analytic, $\lambda_0''(0) < 0$, $\lambda_0(\pi) < 0$ and $\lambda_0'(\nu) \neq 0$ for each $\nu \in (0,\pi)$, the approximation (3.11) yields that there exists some $\varepsilon$-independent constant $\sigma_* > 0$ such that $\mathrm{Re}(\lambda_\varepsilon(\nu)) \leq -\varepsilon^2\sigma_*\nu^2$ holds for all $\nu \in [-\pi,\pi]$. □

If the conditions in Corollary 3.18 are satisfied, then we obtain by Theorem 3.3 nonlinear diffusive stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ as a solution to (2.1) with $\alpha = -\varepsilon^2\lambda_0''(0) + O(\varepsilon^3|\log(\varepsilon)|^5)$.

Regarding instability, Theorems 3.13 and 3.14 readily yield the following result.

**Corollary 3.19.** *If one of the following is true:*

*i. There exists $\gamma_\diamond \in S^1$ and $\lambda_\diamond \in \mathbb{C}$ with $\mathrm{Re}(\lambda_\diamond) > 0$ satisfying $\mathcal{E}_0(\lambda_\diamond,\gamma_\diamond) = 0$;*

*ii. It holds $\lambda_0(\nu) > 0$ for some $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$, where $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$ is given by (3.12) and $\mathcal{N}_\diamond$ is defined in (3.10).*

*Then, provided $\varepsilon > 0$ is sufficiently small, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1) is spectrally unstable.*

Thus, if one of the conditions in Corollary 3.19 is satisfied, then the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinearly unstable against localized and non-localized perturbations by Theorem 3.5.

We emphasize that the conditions in Corollaries 3.18 and 3.19 can be computed with only the singular limit (2.11) of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ as input. More specifically, one needs understanding of the (adjoint) variational equations about the solutions $\psi_\mathrm{h}(x,u_0)$ and $\psi_\mathrm{s}(\check{x})$ to systems (2.5) at $u = u_0$ and (2.6), respectively, and of the eigenvalue problems arising when linearizing equations $v_t = D_2 v_{xx} - G(u_0,v,0)$ and $u_t = D_1 u_{\check{x}\check{x}} - H_1(u,0,0)$ about the stationary solutions $v_\mathrm{h}(x,u_0)$ and $u_\mathrm{s}(\check{x})$, respectively.

### 3.5 Conditions for spectral stability in lower dimensions

In §3.4 we established explicit conditions yielding spectral stability and instability in terms of the eigenvalue problems (3.4), (3.8) and (3.9) and the variational equations (2.9) and (3.6). In this subsection we interpret these results in the case $n = 1$ or $m = 1$. Then, the aforementioned systems become 2-dimensional and we can employ techniques tailored for 2-dimensional linear systems to further simplify the spectral (in)stability conditions in Corollaries 3.18 and 3.19. Throughout this subsection we assume without loss of generality $D_1 = 1$ in the case $m = 1$ and $D_2 = 1$ in the case $n = 1$ – see Remark 2.1.

In the case $n = 1$, the homogeneous fast eigenvalue problem (3.4) becomes 2-dimensional. Thus, Sturm-Liouville theory yields that the root 0 of the fast Evans function $\mathcal{E}_{f,0}$ is always simple and $\mathcal{E}_{f,0}$ has only one (simple) zero $\lambda_*$ of positive real part – see [8, Proposition 4.1].

In the case $m = 1$, the slow variational equation (2.9) becomes 2-dimensional. So, besides the derivative $\psi_s'(\check{x})$, a second, linearly independent solution to (2.9) can be found using Rofe-Beketov's formula – see [4, Chapter 1.9]. We gain explicit control over the evolution of the slow variational equation (2.9) or, equivalently, of the slow eigenvalue problem (3.9) at $\lambda = 0$. Thus, on the one hand, the slow Evans function simplifies to

$$\mathcal{E}_{s,0}(0,\gamma) = \gamma^2 + 2(1 + 2\mathfrak{a}\mathfrak{b})\gamma + 1, \tag{3.14}$$

in the case $m = 1$, where

$$\begin{aligned}
\mathfrak{a} &:= \mathcal{J}'(u_0)\mathcal{J}(u_0) - H_1(u_0,0,0),\\
\mathfrak{b} &:= \mathcal{J}(u_0)\int_0^{\ell_0} \frac{(\partial_u H_1(u_s(\check{x}),0,0)+1)[(u_s'(\check{x}))^2 - (H_1(u_s(\check{x}),0,0))^2]}{[(u_s'(\check{x}))^2 + (H_1(u_s(\check{x}),0,0))^2]^2} d\check{x}\\
&\quad + \frac{H_1(u_0,0,0)}{(\mathcal{J}(u_0))^2 + (H_1(u_0,0,0))^2},
\end{aligned} \tag{3.15}$$

see [8, Propositions 4.4 and 4.7] for more details. On the other hand, we obtain the following result regarding the critical spectral curve attached to the origin.

**Proposition 3.20.** *Let $m = 1$. Suppose that 0 is a simple zero of $\mathcal{E}_{f,0}$. Then, the analytic map $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$, defined in Theorem 3.14, is given by*

$$\lambda_0(\nu) = \mathfrak{a}\mathfrak{w}\frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}\mathfrak{b}}, \tag{3.16}$$

*where $\mathfrak{a}, \mathfrak{b}$ are defined in* (3.15) *and $\mathfrak{w}$ is given by*

$$\mathfrak{w} := -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_h(x,u_0),0)^* \psi_{ad,2}(x) x\, dx}{\int_{-\infty}^{\infty} \psi_{ad,2}(x)^* \partial_x v_h(x,u_0)\, dx}, \tag{3.17}$$

*with $\psi_{ad}(x) = (\psi_{ad,1}(x), \psi_{ad,2}(x))$ a non-trivial, exponentially localized solution to* (3.7)*. If we have in addition $n = 1$, then the expression for $\mathfrak{w}$ simplifies to*

$$\mathfrak{w} = -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_h(x,u_0),0)\partial_x v_h(x,u_0) x\, dx}{\int_{-\infty}^{\infty} (\partial_x v_h(x,u_0))^2\, dx}. \tag{3.18}$$

**Proof.** Besides the derivative $\psi_s'(\check{x}) = (u_s'(\check{x}), H_1(u_s(\check{x}),0,0))$, a second, linearly independent solution to (2.9) is given $(z(\check{x}), z'(\check{x}))$, where

$$z(\check{x}) := u_s'(\check{x}) \int_{\check{x}}^{\ell_0} \frac{(\partial_u H_1(u_s(\check{y}),0,0)+1)[(u_s'(\check{y}))^2 - (H_1(u_s(\check{y}),0,0))^2]}{[(u_s'(\check{y}))^2 + (H_1(u_s(\check{y}),0,0))^2]^2} d\check{y} + \frac{H_1(u_s(\check{x}),0,0)}{(u_s'(\check{x}))^2 + (H_1(u_s(\check{x}),0,0))^2},$$

using Rofe-Beketov's formula. Expressing the evolution $\Phi_s(2\ell_0, 0)$ of (2.9) in terms of $\psi_s'(0)$ and $(z(0), z'(0))$, simplifies the expression for $B(\nu)$ in (3.13) to

$$B(\nu) = -\left[\mathfrak{a} - \begin{pmatrix} 0 & 1 \end{pmatrix}\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0,0)\Upsilon_0\right)^{-1}\begin{pmatrix} 2\mathcal{J}(u_0) \\ 0 \end{pmatrix}\right] = -\left[\mathfrak{a} - \frac{4\mathfrak{a}(1+\mathfrak{a}\mathfrak{b})e^{-i\nu}}{\mathcal{E}_{s,0}(0,e^{-i\nu})}\right]$$

where we use $\mathfrak{b} = z(0)$, $\det\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0,0)\Upsilon_0\right) = e^{2i\nu}\mathcal{E}_{s,0}(0,e^{i\nu}) \neq 0$ and $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ by **(E2)**. By (3.14) it holds $e^{i\nu}\mathcal{E}_{s,0}(0,e^{-i\nu}) = 2(\cos(\nu) + 1 + 2\mathfrak{a}\mathfrak{b})$. Substituting this into the above expression for $B(\nu)$ leads to the desired formula (3.16) for $\lambda_0(\nu) = -\mathfrak{w}B(\nu)$. Finally, in the case $n = 1$, we observe that $(-\partial_x q_h(x,u_0), \partial_x v_h(x,u_0))$ is a solution to equation (3.7) yielding (3.18). This concludes the proof. ◻

The above results lead to the following simplification of the spectral stability conditions in Corollary 3.18 in the lower-dimensional setting.

**Corollary 3.21.** *Suppose $m = 1$ and the following conditions are met:*

*i.* $0$ *is a simple zero of* $\mathcal{E}_{f,0}$*;*

*ii.* $\mathcal{E}_0(\lambda, \gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

*iii. The quantities* $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{w}$*, defined in* (3.15) *and* (3.17)*, have the same (non-zero) sign.*

*Then, provided* $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (2.1) *is spectrally stable.*

*If we have in addition* $n = 1$*, then conditions i. and ii. above are satisfied if and only* $\mathfrak{c}_- = 0$*, with*

$$\mathfrak{c}_\pm := \lim_{R \to \infty} \int_0^{2\pi} \left| \frac{1}{2\pi i} \oint_{\Gamma_R^\pm} \frac{\partial_\lambda \mathcal{E}_{s,0}(\lambda, e^{i\nu})}{\mathcal{E}_{s,0}(\lambda, e^{i\nu})} d\lambda + 1 \right| d\nu \tag{3.19}$$

*where* $\Gamma_R^\pm$ *is the (counter-clockwise) contour in the complex plane consisting of the circle segment* $\{z \in \mathbb{C} : |z \mp R^{-1}| = R, \mathrm{Re}(z) \geq \pm R^{-1}\}$ *and the line joining the points* $iR \pm R^{-1}$ *and* $-iR \pm R^{-1}$*.*

**Proof.** Since we have $\mathfrak{a}\mathfrak{b} > 0$, it holds $\mathcal{E}_{s,0}(0, \gamma) \neq 0$ for each $\gamma \in S^1$ by (3.14). Thus, the first three conditions in Corollary 3.18 are satisfied. Moreover, by Proposition 3.20 we have

$$\lambda_0(\pi) = -\frac{\mathfrak{w}}{\mathfrak{b}}, \qquad \lambda_0''(0) = -\frac{\mathfrak{a}\mathfrak{w}}{2 + 2\mathfrak{a}\mathfrak{b}}, \qquad \lambda_0'(\nu) = -\frac{2\mathfrak{a}(1 + \mathfrak{a}\mathfrak{b})\mathfrak{w}\sin(\nu)}{(1 + 2\mathfrak{a}\mathfrak{b} + \cos(\nu))^2}, \tag{3.20}$$

with $\nu \in \mathbb{R}$. Since $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ are non-zero and have the same sign, the fourth condition in Corollary 3.18 is also satisfied. We conclude that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable.

If $n = 1$, then $0$ is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero $\lambda_*$ of positive real part by [8, Proposition 4.1]. Thus, by Proposition 3.12 the conditions i. and ii. are satisfied if and only if $\mathcal{E}_{s,0}(\lambda, \gamma)$ has precisely one pole of order 1 at $\lambda = \lambda_*$ and no zeros in the closed right half-plane for each $\gamma \in S^1$. The latter is the case if and only if $\mathfrak{c}_- = 0$. □

Thus, in the case $m = n = 1$, we can establish spectral stability by evaluating the four expressions $\mathfrak{a}, \mathfrak{b}, \mathfrak{w}$ and $\mathfrak{c}_-$. In the slowly nonlinear model equation considered in [12] these expressions can be determined exactly – see also [7, Section 3.9]. However, even if these expressions cannot be determined exactly, we expect that spectral stability can be proved using rigorously verified computing. To estimate the errors one needs explicit bounds on the solutions $\psi_{\mathrm{h}}(x, u_0)$ and $\psi_{\mathrm{s}}(\check{x})$ to (2.5) and (2.6) that constitute the singular limit (2.11) and on the functions $H_1, H_2, G$.

On the other hand, the lower-dimensional setting allows us to test for *instability* using parity-type arguments.

**Corollary 3.22.** *Let* $m = n = 1$*. If one of the following is true:*

*i. We have* $\mathfrak{c}_+ \neq 0$*, where* $\mathfrak{c}_+$ *is defined in* (3.19)*;*

*ii. The quantities* $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{w}$*, defined in* (3.15) *and* (3.17)*, are non-zero and have different signs.*

*Then, provided* $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (2.1) *is spectrally unstable.*

**Proof.** First, $0$ is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero $\lambda_*$ of positive real part by [8, Proposition 4.1]. Thus, if $\mathfrak{c}_+ \neq 0$, then there exists by the argument principle a $\gamma \in S^1$, such that either $\mathcal{E}_{s,0}(\cdot, \gamma)$ has no pole at $\lambda_*$ or it has a zero $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$. Thus, it holds either $\mathcal{E}_0(\lambda_*, \gamma) = 0$ or $\mathcal{E}_0(\lambda_0, \gamma) = 0$, which implies by Corollary 3.19 that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable.

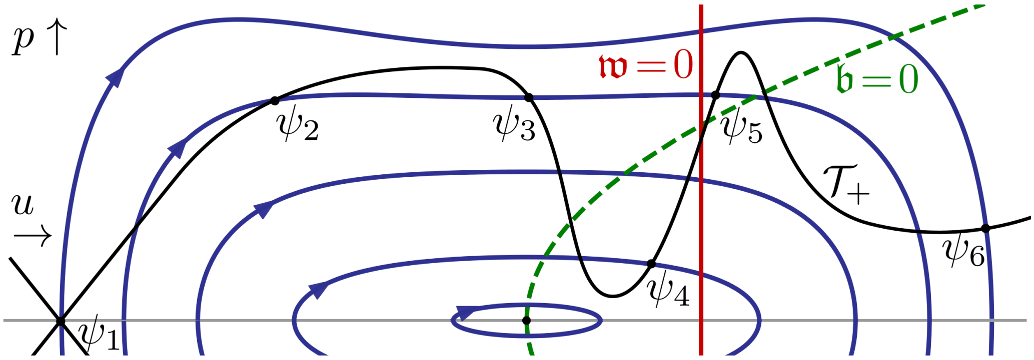



Figure 5: Depicted are five orbits of the slow reduced system (2.6) (in purple). The touch-down curve $\mathcal{T}_+$ intersects these orbits transversally at $\psi_i$, $i = 1, \ldots, 6$. The green dashed line corresponds to the initial values such that $\mathfrak{b} = 0$. We have $\mathfrak{b} > 0$ at $\psi_1, \psi_2, \psi_3$ and $\psi_5$ and $\mathfrak{b} < 0$ at $\psi_4$ and $\psi_6$. The red line corresponds to initial values with $\mathfrak{w} = 0$. We have $\mathfrak{w} > 0$ at $\psi_1, \psi_2$, $\psi_3$ and $\psi_4$ and $\mathfrak{w} < 0$ at $\psi_5$ and $\psi_6$. Finally, we have $\mathfrak{a} < 0$ at $\psi_1$, $\psi_3$ and $\psi_6$ and $\mathfrak{a} > 0$ at $\psi_2$, $\psi_4$ and $\psi_5$. The periodic pulse solutions touching-down at $\psi_1, \psi_3$, $\psi_4$ and $\psi_5$ are spectrally unstable by Corollary 3.22. The solutions touching down at $\psi_2$ and $\psi_6$ are potentially spectrally stable.

Next, suppose the non-zero quantities $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{w}$ have different signs and $1+\mathfrak{a}\mathfrak{b} > 0$. Then, the calculations in (3.20) show that there exists $\nu \in \mathbb{R}$ such that $\lambda_0(\nu) > 0$. So, by Corollary 3.19, $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable. Finally, [8, Proposition 4.7] implies that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable in the case $1 + \mathfrak{a}\mathfrak{b} \leq 0$. □

**Remark 3.23.** In the case $m = 1$, the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ determine by Proposition 3.20 and (3.14) the spectral configuration about the origin and play an important role in destabilization processes – see §4. We elaborate on the geometric interpretation of these quantities. As mentioned in §2.2, the quantity $\mathfrak{a}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution $\psi_{\mathrm{s}}$ to (2.6) at $\psi_{\mathrm{s}}(0) = (u_0, \mathcal{J}(u_0))$ and, by symmetry, between the take-off curve $\mathcal{T}_-$ and $\psi_{\mathrm{s}}$ at $\psi_{\mathrm{s}}(2\ell_0) = R_s\psi_{\mathrm{s}}(0)$ – see Figure 5. If $\mathfrak{a} = 0$, then $\psi_{\mathrm{s}}(\check{x})$ is tangent to the touch-down curve at $\check{x} = 0$.

The quantity $\mathfrak{b}$ depends on the dynamics in the slow reduced system (2.6) only. Since $\psi_{\mathrm{s}}(\ell_0)$ is contained in $\ker(I - R_s)$ by assumption **(E2)**, the vector $\psi_\diamond = (H_1(u_{\mathrm{s}}(\ell_0)), 0, 0)^{-1}, 0)$ is a normal to the tangent space of the curve $\psi_{\mathrm{s}}(\check{x})$ at $\check{x} = \ell_0$ such that $\det(\psi_\diamond \mid \psi_{\mathrm{s}}'(\ell_0)) = 1$. Tracking the tangent space along the flow of (2.6) we obtain the solution $\Phi_{\mathrm{s}}(\check{x}, \ell_0)\psi_\diamond = (z(\check{x}), z'(\check{x}))$ to (2.9). We have that $z(0)$ equals the quantity $\mathfrak{b}$.

Observe that $z(\check{x})$ has precisely one root between two consecutive zeros of $u_{\mathrm{s}}'(\check{x})$, since the derivative of $u_{\mathrm{s}}'(\check{x})/z(\check{x})$ never vanishes between two zeros of $u_{\mathrm{s}}'$. Therefore, given that the orbit of $\psi_{\mathrm{s}}$ in the slow reduced system (2.6) crosses the line $p = 0$ at $u = u_\pm$ with $u_- < u_+$, there is precisely one initial value $u_0 = u_{\mathrm{s}}(0) \in (u_-, u_+)$ for which $\mathfrak{b} = 0$ – see Figure 5.

The quantity $\mathfrak{w}$ occurs in [37], where one derives asymptotic interaction laws for quasi-stationary pulse solutions to models of the form (2.1). More precisely, one establishes in [37] an ODE, which describes the (leading-order) evolution of the pulse locations over time, assuming existence and smoothness of the quasi-stationary pulse pattern. The pulse locations of our *stationary*, periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ to (2.1) correspond naturally to an equilibrium of this ODE. The quantity $\mathfrak{w}$ occurs as a factor in the linearization of the ODE about this equilibrium – see [37, Section 6.2.1]. Thus, the sign of $\mathfrak{w}$ corresponds to the character of the equilibrium. Loosely speaking, $\mathfrak{w}$ measures the stability of $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ against perturbations of the pulse locations. This relates to the fact that vanishing of $\mathfrak{w}$ corresponds to a transition of the critical spectral curve through the imaginary axis – see §4. ■

**Remark 3.24.** In [47] the spectral stability of stationary, spatially periodic pulse solutions is studied in the generalized Gierer-Meinhardt equation

$$\begin{aligned} \varepsilon^2 u_t &= u_{xx} - \varepsilon^2 \mu u + \varepsilon u^{\alpha_1} v^{\beta_1}, \\ v_t &= v_{xx} - v + u^{\alpha_2} v^{\beta_2}, \end{aligned} \qquad (u,v) \in \mathbb{R}^2, x \in \mathbb{R}, \tag{3.21}$$

with parameters $\alpha_1 \in \mathbb{R}, \alpha_2 < 0, \beta_{1,2} \in \mathbb{Z}_{>1}$ and $\mu > 0$ satisfying

$$(\alpha_1 - 1)(\beta_2 - 1) - \alpha_2 \beta_1 > 0.$$

It is not difficult to verify that assumptions **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true for (3.21). Thus, Theorem 2.2 provides a periodic pulse solution to (3.21). The slow variational equation (2.9) corresponds in this setting to the autonomous equation $u_{\check{x}\check{x}} = \mu u$. The $v$-component of the homoclinic solution $\psi_{\mathrm{h}}(x, u_0)$ to system (2.5) at $u = u_0$ is given by

$$v_{\mathrm{h}}(x, u_0) = u_0^{-\frac{\alpha_2}{\beta_2 - 1}} w_{\mathrm{h}}(x), \quad w_{\mathrm{h}}(x) := \left(\frac{\beta_2 + 1}{2} \operatorname{sech}^2\left(\frac{(\beta_2 - 1)\, x}{2}\right)\right)^{\frac{1}{\beta_2 - 1}}.$$

Thus, using integration by parts, we calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ in Proposition 3.20

$$\mathfrak{a} = \mathcal{J}(u_0)\mathcal{J}'(u_0) - \mu u_0, \quad \mathfrak{b} = \frac{\cosh^2\left(\ell_0 \sqrt{\mu}\right)}{4\mu u_0}, \quad \mathfrak{w} = -\frac{\alpha_2 \int_{-\infty}^{\infty} w_{\mathrm{h}}(x)^{\beta_2 + 1} dx}{u_0\, (\beta_2 + 1) \int_{-\infty}^{\infty} \left(w_{\mathrm{h}}'(x)\right)^2 dx},$$

where $\mathcal{J} \colon (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{J}(u) = u^{\alpha_1 - \frac{\alpha_2 \beta_1}{\beta_2 - 1}} \int_0^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx.$$

It holds $\mathfrak{b}\mathfrak{w} > 0$, since we have $\beta_{1,2} > 1, \alpha_2 < 0$ and $\mu > 0$. In addition, the signs of $\mathfrak{a}\mathfrak{w}$ and $\mathfrak{a}\mathfrak{b}$ are equal to the sign of

$$\frac{\mathfrak{a}}{u_0} = \left(\alpha_1 - \frac{\alpha_2 \beta_1}{\beta_2 - 1}\right)\left(u_0^{\alpha_1 - \frac{\alpha_2 \beta_1}{\beta_2 - 1} - 1} \int_0^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx\right)^2 - \mu.$$

Thus, the sign of $\frac{\mathfrak{a}}{u_0}$ determines whether condition iii. in Corollary 3.21 is satisfied. The quantity $\frac{\mathfrak{a}}{u_0}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution $\psi_{\mathrm{s}}$ – see Remark 3.23.

One can verify that the leading-order expression (3.16) of the critical spectral curve coincides with the one in [47] derived with a different method – see also Remark 1.1. ■

# 4 Destabilization mechanisms

In this section we focus on instabilities of periodic pulse solutions to (2.1) as system parameters are varied. Let $\check{\phi}_{\mathrm{p},\varepsilon}$ be a periodic pulse solution to (2.1), established in Theorem 2.2. To describe the spectral geometry as $\check{\phi}_{\mathrm{p},\varepsilon}$ destabilizes, we need as much analytical grip as possible. Therefore, we restrict ourselves to the case $m = n = 1$ – see §3.5. We assume that equation (2.1) depends on a real parameter $\mu$. The periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable if the three conditions in Corollary 3.8 are satisfied. A codimension-one instability of $\check{\phi}_{\mathrm{p},\varepsilon}$ occurs if one of these conditions fails as we vary $\mu$, while the others are still valid. Denote by $\mathcal{E}_{\varepsilon,\mu}(\lambda, \gamma)$ the associated Evans function (depending on $\mu$). Suppose one of the conditions in Corollary 3.8 is violated by a pair $(\lambda_*, \gamma_*) \in i\mathbb{R} \times S^1$ at $\mu = \mu_*$, where $\gamma_* = e^{i\nu_*}$ for some $\nu_* \in \mathbb{R}$. Consequently, it holds $\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*) = 0$. If we have $\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*) \neq 0$, the implicit function theorem yields a local expansion of the marginally stable spectral curve $\lambda_c(\nu)$ through $\lambda_*$:

$$\lambda_c(\nu) = \lambda_* + \frac{a_2}{2!}(\nu - \nu_*)^2 + \frac{a_4}{4!}(\nu - \nu_*)^4 + O\left((\nu - \nu_*)^6\right),$$

with $a_2, a_4 \in \mathbb{C}$. Note that Proposition 3.7 implies that the odd coefficients in the expansion of $\lambda_c(\nu)$ must be zero. The leading coefficient $a_2$ can be computed through implicit differentiation:

$$a_2 = \frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)\gamma_*^2}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)}.$$

In the case $a_2 = 0$, we have

$$a_4 = \frac{-\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)\gamma_*^4}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)}.$$

This gives rise to the following classification of codimension-one instabilities – see [38, Section 3.3].

- *$\gamma_*$-Hopf.* The second and third condition in Corollary 3.8 are satisfied and the first condition is violated by a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm 1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)\gamma_*^2}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, \gamma_*)}\right] \neq 0.$$

- *Spatial period doubling.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at $\gamma = -1$ so that

$$\mathcal{E}_{\varepsilon,\mu_*}(0,-1) = 0, \quad \partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(0,-1)\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,-1) < 0, \quad \partial_\mu \mathcal{E}_{\varepsilon,\mu_*}(0,-1) \neq 0.$$

- *$\gamma_*$-Turing.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at a unique pair $\gamma_*^{\pm 1} \in S^1 \setminus \{\pm 1\}$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*) = 0, \quad \partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*)\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*)\gamma_*^2 < 0, \quad \partial_\mu \mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*) \neq 0.$$

- *Sideband.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1) = 0, \quad \partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(0,1)\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1) > 0, \quad \partial_{\gamma\gamma\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1) \neq 0.$$

- *Fold/Pitchfork.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(0,1) = 0, \quad \partial_{\lambda\lambda}\mathcal{E}_{\varepsilon,\mu_*}(0,1), \partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1), \partial_{\lambda\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1) \neq 0.$$

Using the results from §3 one easily verifies that the only possible primary codimension-one instabilities are of sideband, Hopf or period doubling type.

**Proposition 4.1.** *Suppose $m = n = 1$. The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ to* (2.1) *cannot be destabilized through a Turing or fold instability.*

**Proof.** In the case of a $\gamma_*$-Turing instability, $\mathcal{E}_{\varepsilon,\mu_*}(0,\cdot)$ has double roots $\gamma_*^{\pm 1}$ and 1 with $\gamma_* \in S^1 \setminus \{1\}$. However, this is impossible, since $\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma)$ is a quartic polynomial in $\gamma$ by Proposition 3.12. In the case of a fold instability, 0 is a double root of the reduced Evans function $\mathcal{E}_{0,\mu_*}(\cdot, 1)$ by Theorem 3.13. Since 0 is a simple root of the fast Evans function $\mathcal{E}_{f,0,\mu_*}$ by [8, Proposition 4.1], the slow Evans function $\mathcal{E}_{s,0,\mu_*}(\cdot, 1)$ also has a root 0. Thus, identity (3.14) yields $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*) = -1$. So, by Corollary 3.22 there exists a $\lambda$ in the spectrum $\sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda) > 0$. Hence, the first condition in Corollary 3.8 is not satisfied, which contradicts the occurrence of a fold instability. □

To identify which one of the three remaining instabilities occurs when the periodic pulse $\check{\phi}_{p,\varepsilon}$ destabilizes, does not require control over the full Evans function $\mathcal{E}_{\varepsilon,\mu}$. It is sufficient to track the quantities $\mathfrak{a}(\mu)$, $\mathfrak{b}(\mu)$ and $\mathfrak{w}(\mu)$ and the roots of the slow Evans function $\mathcal{E}_{s,0,\mu}$ as we vary $\mu$. Indeed, the zeros of the fast Evans function $\mathcal{E}_{f,0,\mu}$ will in general depend on the parameter $\mu$. However, by [8, Proposition 4.1] the relative position of these zeros with respect to the origin is fixed, i.e. no root of the fast Evans function can pass through the origin as we vary $\mu$. Thus, according to Theorems 3.13 and 3.14, generic instabilities occur if either the curve $\lambda_{0,\mu}(\nu)$, that is attached to the origin, or a curve $\lambda_{*,\mu}(\nu)$ satisfying $\mathcal{E}_{s,0,\mu}(\lambda_{*,\mu}(\nu), e^{i\nu}) = 0$ transits through the imaginary axis as we vary $\mu$. By identity (3.14) and Proposition 3.20 this is precisely the case if either one of the quantities $\mathfrak{a}(\mu)$, $\mathfrak{b}(\mu)$ or $\mathfrak{w}(\mu)$, defined in (3.15) and (3.17), changes sign or, for some $\gamma_* \in S^1$, there is a complex conjugate pair of roots of the slow Evans function $\mathcal{E}_{s,0,\mu}(\cdot, \gamma_*)$ moving through the imaginary axis $i\mathbb{R} \setminus \{0\}$ as $\mu$ passes through some value $\mu_*$. Thus, we distinguish between the following generic destabilization scenarios:

**(D1)** $\mathfrak{w}(\mu_*) = 0$, $\partial_\mu \mathfrak{w}(\mu_*) \neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*) > 0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $\gamma \in S^1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$;

**(D2)** $\mathfrak{b}(\mu_*) = 0$, $\partial_\mu \mathfrak{b}(\mu_*) \neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{w}(\mu_*) > 0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $\gamma \in S^1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \neq 0$;

**(D3)** $\mathfrak{a}(\mu_*) = 0$, $\partial_\mu \mathfrak{a}(\mu_*) \neq 0$, $\mathfrak{b}(\mu_*)\mathfrak{w}(\mu_*) > 0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $\gamma \in S^1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \neq 0$;

**(D4)** There is a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm 1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{s,0,\mu_*}(\lambda_*, \gamma_*) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{s,0,\mu_*}(\lambda_*, \gamma_*)\gamma_*^2}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\lambda_*, \gamma_*)}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{s,0,\mu_*}(\lambda_*, \gamma_*)}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\lambda_*, \gamma_*)}\right] \neq 0.$$

In addition, $\mathfrak{a}(\mu_*)$, $\mathfrak{b}(\mu_*)$ and $\mathfrak{w}(\mu_*)$ have the same non-zero sign and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $(\lambda, \gamma) \in S^1 \times \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $(\lambda, \gamma) \neq (\pm\lambda_*, \gamma_*^{\pm 1})$.

We identify the type of instability occurring in these four scenarios. Clearly, the following result is an immediate consequence of Theorems 3.13 and 3.14 and Proposition 3.20.

**Corollary 4.2.** *Assume $m = n = 1$ and **(D4)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{p,\varepsilon}$ to* (2.1) *destabilizes through a $\gamma_\varepsilon$-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon - \gamma_*| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

The remainder of this section is devoted to the identification of the type of instability occurring in the three other scenarios, which requires detailed control over the spectral geometry about the origin.

**Remark 4.3.** Destabilization scenario **(D4)** has been studied in great detail when periodic pulse solutions approach a homoclinic limit. While decreasing the wave number $k$, the character of destabilization alternates between $+1$-Hopf and $-1$-Hopf instabilities. In $(k, \mu)$-space the curves $\mathcal{H}_{\pm 1}$ corresponding to $\pm 1$-Hopf instabilities intersect infinitely often as they oscillate about each other while both converging to the Hopf destabilization point of the homoclinic limit solution on the line $k = 0$. This phenomenon is called the *Hopf dance*. In the limit $\varepsilon \to 0$ the two curves $\mathcal{H}_{\pm 1}$ cover the boundary of the region of stable pulse solutions. The boundary is non-smooth at the (transversal) intersection points of $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$. This corresponds to an associated higher order phenomenon: *the belly dance*. The non-smoothness of the stability boundary persists for sufficiently small $\varepsilon > 0$. The Hopf and belly dance were first analytically observed in the Gierer-Meinhardt system [13, 47]. Recently, it is shown in [12] that these phenomena are persistent mechanisms that occur in the general class (2.1) of slowly nonlinear systems. ■

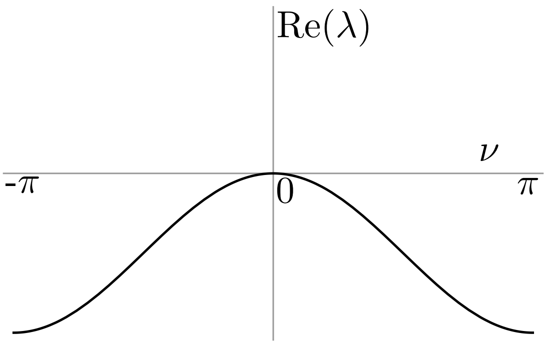
## 4.1 The first destabilization scenario

Let **(D1)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) < 0$. There exists a neighborhood $M \subset \mathbb{R}$ of $\mu_*$ such that it holds $\mathcal{E}_{s,0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$. Moreover, 0 is a simple zero of $\mathcal{E}_{f,0,\mu}$ by [8, Proposition 4.1]. Thus, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum for any $\mu \in M$ by Theorem 3.13. In addition, $\lambda_{\varepsilon,\mu}$ is real-valued and analytic and, by Proposition 3.20, we have the leading-order approximation

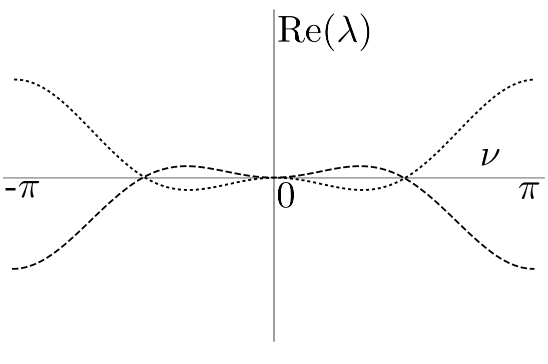
$$\lambda_{\varepsilon,\mu}(\nu) = \varepsilon^2 \mathfrak{a}(\mu)\mathfrak{w}(\mu) \frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}(\mu)\mathfrak{b}(\mu)} + O\left(\varepsilon^3 \left|\log(\varepsilon)\right|^5\right), \tag{4.1}$$

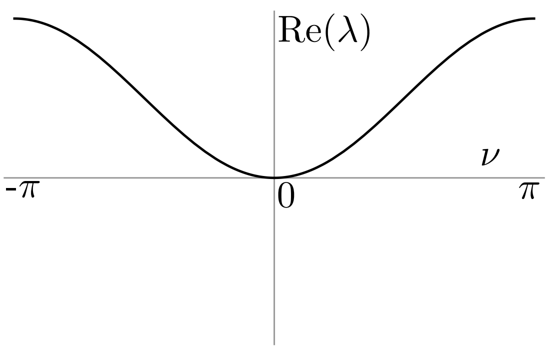
for any $\mu \in M$ and $\nu \in \mathbb{R}$. So, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, for $\mu \in M$ with $|\mu - \mu_*| > \delta$ the approximation (4.1) gives the spectral configuration depicted in Figures 6a and 6c. Hence, $\check{\phi}_{p,\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and unstable for $\mu > \mu_* + \delta$. For $|\mu - \mu_*| \leq \delta$ our leading-order approximation (4.1) is insufficient to determine the precise position of the critical spectral curve with respect to the imaginary axis. However, since $\lambda_{\varepsilon,\mu}$ is real-valued for any $\mu \in M$ and Turing instabilities do not occur by Proposition 4.1, we have obtained the following result.

**Proposition 4.4.** *Assume $m = n = 1$ and **(D1)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{p,\varepsilon}$ to* (2.1) *destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*




(a) $\mu < \mu_* - \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) > 0$




(b) $|\mu - \mu_*| \leq \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) \approx 0$

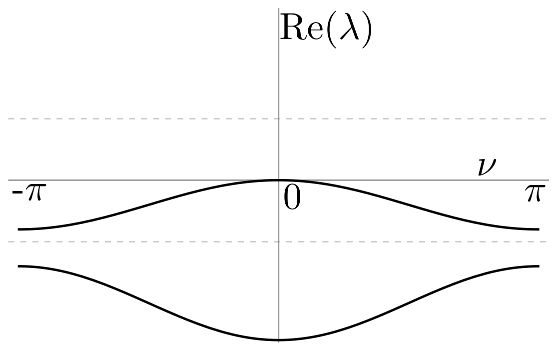
(c) $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) < 0$

Figure 6: The spectral geometry about the origin in the first generic destabilization scenario **(D1)** with $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) < 0$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

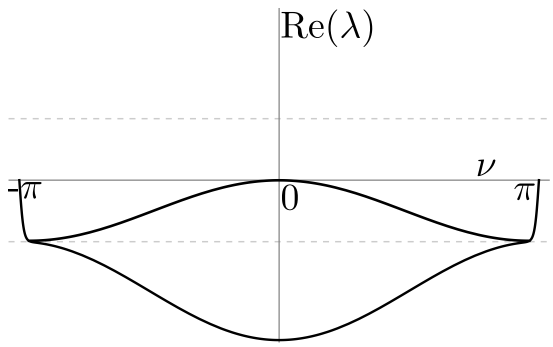
## 4.2 The second destabilization scenario

Let **(D2)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu\mathfrak{b}(\mu_*) < 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of $\mu_*$ such that $\mathfrak{a}(\mu)\mathfrak{w}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0, \delta)$ with $\mathrm{Re}(\lambda) \geq 0$. In addition, it holds $\mathcal{E}_{s,0,\mu}(0, \gamma) \neq 0$ for any $\gamma \in S^1$ and $\mu \in M$ with $\mu < \mu_* - \delta$ by Proposition 3.14. So, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum for any $\mu \in M$ with $\mu < \mu_* - \delta$ by Theorem 3.13. In that situation $\lambda_{\varepsilon,\mu}(\nu)$ is by Proposition 3.20 approximated by (4.1) – see Figure 7a. Denote

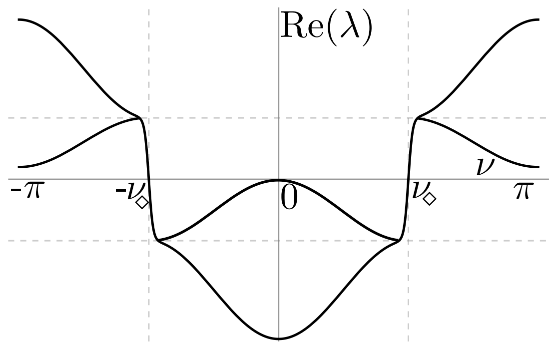
$$\nu_\diamond(\mu) := \arccos\left(\max\{-1 - 2\mathfrak{a}(\mu)\mathfrak{b}(\mu), -1\}\right), \quad \mu \in M.$$



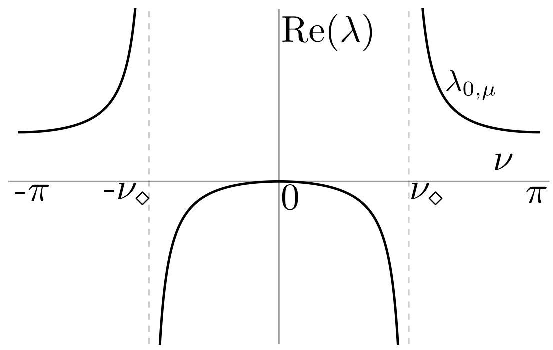

(a) $\mu < \mu_* - \delta$, $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$



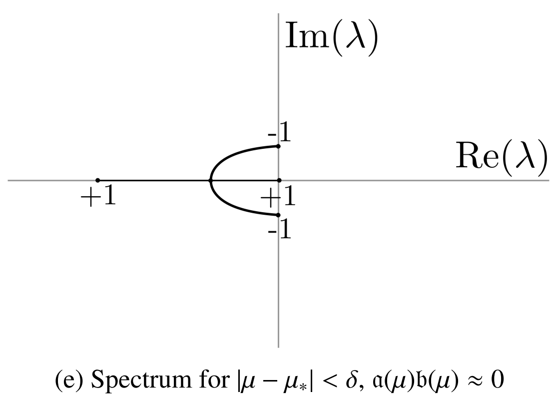

(b) $|\mu - \mu_*| < \delta$, $\mathfrak{a}(\mu)\mathfrak{b}(\mu) \approx 0$



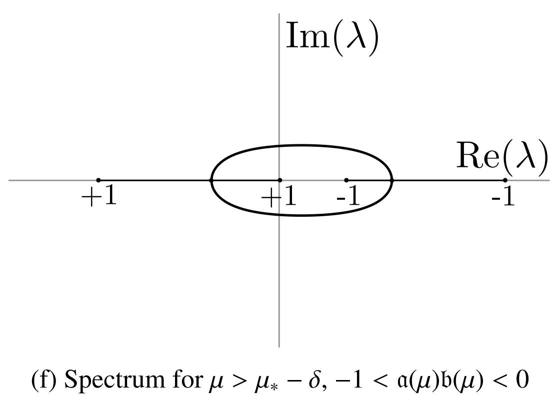

(c) $\mu > \mu_* + \delta$, $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$




(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, and $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$ with singularities $\pm\nu_\diamond(\mu)$




(e) Spectrum for $|\mu - \mu_*| < \delta$, $\mathfrak{a}(\mu)\mathfrak{b}(\mu) \approx 0$




(f) Spectrum for $\mu > \mu_* - \delta$, $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$

Figure 7: The spectral geometry about the origin in the second destabilization scenario **(D2)** with $\mathfrak{a}(\mu_*)\partial_\mu\mathfrak{b}(\mu_*) < 0$. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = \mathcal{O}(\varepsilon^2)$.

For any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists by Theorem 3.14 and Proposition 3.20 a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0,\delta)$ that is approximated by (4.1) – see Figure 7d. Combining this with Proposition 3.14 implies that for any $\mu \in M$ and $\nu \in [-\pi, \pi]$ there are precisely two $e^{i\nu}$-eigenvalues of positive real part if $|\nu| > \nu_\diamond(\mu) + \delta$ and no $e^{i\nu}$-eigenvalues of positive real part if $|\nu| < \nu_\diamond(\mu) - \delta$ – see Figure 7c.

Therefore, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and there is unstable spectrum for $\mu > \mu_* + \delta$. In particular, we observe that $e^{i\nu}$-eigenvalues with $|\nu \pm \pi| < \delta$ are in the right half-plane strictly before $e^{i\nu}$-eigenvalue with $|\nu| < \delta$ as $\mu$ increases. Thus, a sideband instability cannot occur.

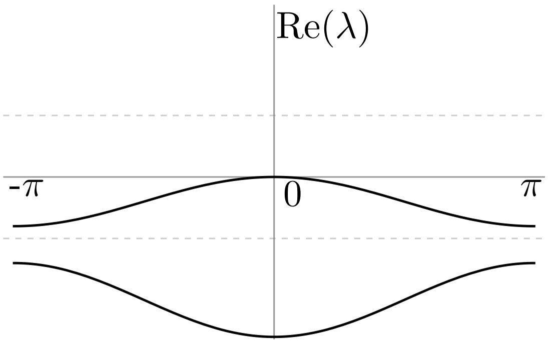
Now suppose a spatial period doubling bifurcation occurs at $\mu = \mu_\varepsilon$. By the previous observations there are precisely two $-1$-eigenvalues in the right half-plane for $\mu \in M$ with $\mu > \mu_* + \delta \geq \mu_\varepsilon$. By definition of a period doubling bifurcation, the most unstable one of these $-1$-eigenvalues must have crossed the imaginary axis at the origin. Since the spectrum is symmetric in the real axis – see Proposition 3.7 – the same holds for the other $-1$-eigenvalue. If the $-1$-eigenvalues cross simultaneously, then $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ has a root 1 of multiplicity two and a root $-1$ of multiplicity four, which is impossible, since $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ is a quartic polynomial by Proposition 3.12. If one $-1$-eigenvalue crosses first, then, by the implicit function theorem and symmetry of the spectrum in the real axis, this $-1$-eigenvalue is attached to a spectral branch that lies on the real axis. So, if the second $-1$-eigenvalue crosses at $\mu = \tilde{\mu}_\varepsilon > \mu_\varepsilon$, then $\mathcal{E}_{\varepsilon,\tilde{\mu}_\varepsilon}(0,\cdot)$ has double roots 1 and $-1$ and simple roots $\gamma^{\pm 1}$ for some $\gamma \in S^1 \setminus \{\pm 1\}$, which is again impossible. We conclude that a period doubling bifurcation cannot occur. So, by Proposition 4.1 a Hopf instability occurs – see Figure 7b. Thus, we obtain the following result.

**Proposition 4.5.** *Assume $m = n = 1$ and **(D2)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1) destabilizes through a $\gamma_\varepsilon$-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon + 1| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

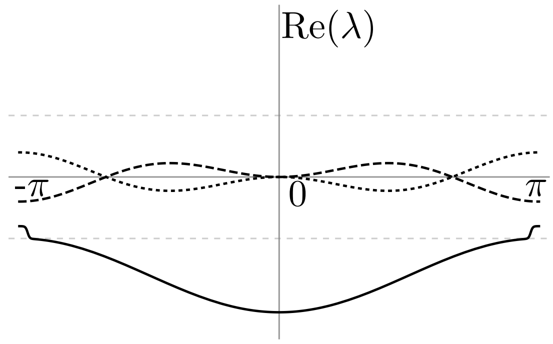
### 4.3 The third destabilization scenario

Let **(D3)** hold true and assume without loss of generality $\mathfrak{w}(\mu_*)\partial_\mu \mathfrak{a}(\mu_*) < 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of $\mu_*$ such that $\mathfrak{w}(\mu)\mathfrak{b}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0,\delta)$ with $\mathrm{Re}(\lambda) \geq 0$. As in the second destabilization scenario **(D2)**, for any $\mu \in M$ with $\mu < \mu_* - \delta$, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum and it is approximated by (4.1) – see Figure 8a. Also similar to scenario **(D2)**, we establish that for any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0,\delta)$ that is approximated by (4.1) – see Figure 8d. Combining this with Proposition 3.14 implies that for any $\mu \in M$ with $\mu > \mu_* + \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there is precisely one $e^{i\nu}$-eigenvalue of positive real part. This excludes the possibility of a Hopf destabilization. So, by Proposition 4.1 either a sideband instability or period doubling bifurcation occurs – see Figure 8. Thus, we obtain the following result.

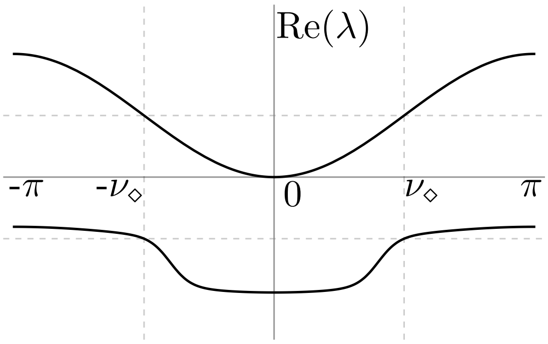
**Proposition 4.6.** *Assume $m = n = 1$ and **(D3)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (2.1) destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*



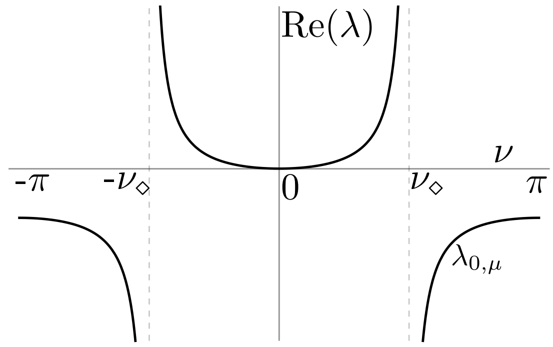

(a) $\mu < \mu_* - \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) > 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$




(b) $|\mu - \mu_*| < \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) \approx 0$ and $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$




(c) $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) < 0$ and $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$




(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu)\mathfrak{w}(\mu) < 0$ and $-1 < \mathfrak{a}(\mu)\mathfrak{b}(\mu) < 0$ with singularities $\pm\nu_\diamond(\mu)$

Figure 8: The spectral geometry about the origin in the third destabilization scenario **(D3)** with $\mathfrak{w}(\mu_*)\partial_\mu \mathfrak{a}(\mu_*) < 0$. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = O(\varepsilon^2)$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

# 5 Proof of the main result

## 5.1 Set-up

In this section we prove Theorem 3.14. Thus, we assume 0 is a simple zero of the fast Evans function $\mathcal{E}_{f,0}$. Moreover, we take $\delta > 0$ and denote

$$\mathcal{N}_\diamond = \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta = \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta).$$

For each $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$ the reduced Evans function $\mathcal{E}_0(\cdot, e^{i\nu})$ has a simple root at 0 by Proposition 3.12. Since $\mathcal{E}_0$ is analytic on its domain, there exists $\varsigma > 0$ such that there are no other roots of $\mathcal{E}_0(\cdot, e^{i\nu})$ in the closed ball $\overline{B(0, \varsigma)}$ for any $\nu \in \mathcal{S}_\delta$. So, provided $\varepsilon > 0$ is sufficiently small, there exists by Theorem 3.13 a unique (simple) root $\lambda_\varepsilon(\nu)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ in $B(0, \varsigma)$ for each $\nu \in \mathcal{S}_\delta$. By Proposition 3.7, the function $\lambda_\varepsilon \colon \mathcal{S}_\delta \to B(0, \varsigma)$ is real-valued, $2\pi$-periodic and even. Moreover, since $\mathcal{E}_\varepsilon$ is analytic, $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ is also analytic by the implicit function theorem. By translational invariance, it holds $\lambda_\varepsilon(0) = 0$ if we have $0 \in \mathcal{S}_\delta$. Thus, all that remains to prove Theorem 3.14 is to establish the approximation (3.11).

We describe our approach to obtain the desired approximation. Fix $\nu \in \mathcal{S}_\delta$. On the one hand, since we have $\mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{i\nu}) = 0$, the full eigenvalue problem (3.2) admits at $\lambda = \lambda_\varepsilon(\nu)$ a solution $\tilde{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$, which satisfies $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(x + 2L_\varepsilon)$ for each $x \in \mathbb{R}$. On the other hand, the derivative $\phi'_{\mathrm{p},\varepsilon}(x)$ of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), p_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x), q_{\mathrm{p},\varepsilon}(x))$

to (2.3) is a solution to (3.2) at $\lambda = 0$. Therefore

$$\psi_{\nu,\varepsilon}(x) := \begin{pmatrix} \tilde{v}_{\nu,\varepsilon}(x) - v'_{\mathrm{p},\varepsilon}(x) \\ \tilde{q}_{\nu,\varepsilon}(x) - q'_{\mathrm{p},\varepsilon}(x) \end{pmatrix},$$

solves the inhomogeneous problem

$$\psi_x = \mathcal{A}_f(x)\psi + \begin{pmatrix} 0 \\ \mathcal{B}_{\nu,\varepsilon}(x) + \lambda_\varepsilon(\nu)\tilde{v}_{\nu,\varepsilon}(x) \end{pmatrix}, \quad \psi \in \mathbb{C}^{2n}, \tag{5.1}$$

where $\mathcal{A}_f(x)$ is the coefficient matrix of the fast variational equation (3.6) and $B_{\nu,\varepsilon}(x)$ is given by

$$\mathcal{B}_{\nu,\varepsilon}(x) := \begin{pmatrix} \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x, u_0), 0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \tilde{u}_{\nu,\varepsilon}(x) - u'_{p,\varepsilon}(x) \\ \tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x) \end{pmatrix},$$

with $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$. The operator $\mathcal{L}_f$ associated with the fast variational equation (3.6) is Fredholm of index 0 by Proposition 3.10. In addition, by Remark 3.11 there exists a non-trivial, exponentially localized solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint problem (3.7), which is unique up to scalar multiples. Thus, applying the solvability condition in [35, Lemma 4.2] to equation (5.1) leads to the key identity

$$\lambda_\varepsilon(\nu) \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x) dx = -\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \mathcal{B}_{\nu,\varepsilon}(x) dx. \tag{5.2}$$

Hence, to obtain a leading-order expression of $\lambda_\varepsilon(\nu)$, it is sufficient to approximate the two integrals in (5.2). Thus, we need leading-order expressions of the solution $\tilde{\varphi}_{\nu,\varepsilon}(x)$ to (3.2), of the solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ to (2.2) and of the difference $\tilde{\varphi}_{\nu,\varepsilon}(x) - \phi'_{\mathrm{p},\varepsilon}(x)$. Clearly, we can approximate $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ by its singular limit – see Theorem 2.2. To obtain leading-order expressions for the other quantities, we proceed as follows. Define

$$D_{\eta,\varepsilon} := \{\lambda \in \mathbb{C} : |\lambda||\log(\varepsilon)| < \eta\}, \tag{5.3}$$

with $\eta > 0$ an $\varepsilon$-independent constant. Moreover, consider the intervals

$$I_{f,\varepsilon} := [-\Xi_\varepsilon, \Xi_\varepsilon], \quad I_{s,\varepsilon} := [\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \quad \Xi_\varepsilon := -\frac{8\log(\varepsilon)}{\min\{\mu_0, \mu_r, \mu_{\mathrm{h}}\}}, \tag{5.4}$$

with $\mu_{\mathrm{h}} > 0$ as in (2.8), $\mu_0 > 0$ as in Theorem 2.2 and $\mu_r > 0$ as in Lemma 3.9. For any $\nu \in \mathcal{S}_\delta$ and $\lambda \in D_{\eta,\varepsilon}$ we establish a *piecewise continuous* solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.2) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$ – see Figure 9. We explicitly construct $\varphi_{\nu,\varepsilon}$ via Lin's method [5, 30, 48] using the singular limit structure (2.11) of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}$ as our framework.

By Theorem 2.2, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{f,\varepsilon}$ approximated by the pulse solution $\phi_{\mathrm{h}}(x, u_0)$ to the fast reduced system (2.4). Moreover, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{s,\varepsilon}$ approximated by the solution $(\psi_{\mathrm{s}}(\varepsilon x), 0)$ on the slow manifold, where $\psi_{\mathrm{s}}$ solves the slow reduced system (2.6). The endpoints of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}$ correspond to the $x$-values for which $\phi_{\mathrm{p},\varepsilon}(x)$ converges to one of the two non-smooth corners $(u_0, \pm\mathcal{J}(u_0), 0, 0)$ of the singular concatenation (2.11) as $\varepsilon \to 0$.

For $x \in I_{f,\varepsilon}$, we establish a reduced eigenvalue problem by setting $\varepsilon$ and $\lambda$ to 0 in (3.2), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x, u_0)$. The reduced eigenvalue problem admits exponential trichotomies on both half-lines – see §A. Hence, one can construct solutions to (3.2) for $\lambda \in D_{\eta,\varepsilon}$ using variation of constants formulas on the intervals

$$I^-_{f,\varepsilon} := [-\Xi_\varepsilon, 0], \quad I^+_{f,\varepsilon} := [0, \Xi_\varepsilon]. \tag{5.5}$$

We can control the perturbation terms in these formulas by taking $\eta, \varepsilon > 0$ sufficiently small.

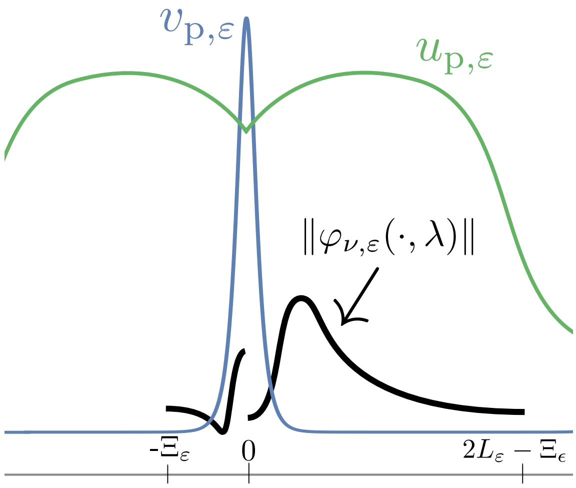



Figure 9: A sketch of the piecewise continuous eigenfunction $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ on its domain of definition $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$. Also depicted are the $u$- and $v$-component of the periodic pulse solution $\hat{\phi}_{\mathrm{p},\varepsilon}$ (in the case $n = m = 1$).

For $x \in I_{s,\varepsilon}$, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in (3.2) is exponentially small by assumption **(S1)** and Theorem 2.2. Thus, we obtain a reduced eigenvalue problem by setting $\mathcal{A}_{21,\varepsilon}(x)$ to 0 in (3.2), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by $(\psi_{\mathrm{s}}(\varepsilon x), 0)$. The reduced eigenvalue problem is upper-triangular and the spectrum of the lower-right block has a consistent splitting into $n$ unstable and $n$ stable eigenvalues – see Lemma 3.9. This splitting yields the existence of an exponential trichotomy on the interval $I_{s,\varepsilon}$. Thus, one can construct solutions to (3.2) on $I_{s,\varepsilon}$ using the variation of constants formula again.

In summary, we obtain variation of constants formulas for solutions to (3.2) on the three intervals $I_{f,\varepsilon}^+$, $I_{f,\varepsilon}^-$ and $I_{s,\varepsilon}$. Matching of these expressions yields for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.2) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$ which has a jump at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$. We show that for any $\nu \in \mathcal{S}_\delta$ the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes at a unique $\lambda$-value $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$. Thus, since (3.2) is $2L_\varepsilon$-periodic, there exists a continuous solution $\check{\varphi}_{\nu,\varepsilon}$ to (3.2) at $\lambda = \tilde{\lambda}_\varepsilon(\nu)$ satisfying

$$\begin{aligned} \check{\varphi}_{\nu,\varepsilon}(x) &= \varphi_{\nu,\varepsilon}(x, \tilde{\lambda}_\varepsilon(\nu)), \qquad x \in I_{f,\varepsilon} \cup I_{s,\varepsilon}, \\ \check{\varphi}_{\nu,\varepsilon}(x) &= e^{i\nu}\check{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x), \qquad x \in \mathbb{R}, \end{aligned} \qquad \nu \in \mathcal{S}_\delta. \tag{5.6}$$

Consequently, $\tilde{\lambda}_\varepsilon(\nu)$ must be a zero of the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$. Since the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ has a unique root $\lambda_\varepsilon(\nu)$ in $B(0,\varsigma)$, we must have $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ for each $\nu \in \mathcal{S}_\delta$. Since the key identity (5.2) is satisfied for any solution $\tilde{\varphi}_{\nu,\varepsilon}$ to (3.2) at $\lambda = \lambda_\varepsilon(\nu)$ satisfying $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x)$ for any $x \in \mathbb{R}$, it holds in particular for $\tilde{\varphi}_{\nu,\varepsilon} = \check{\varphi}_{\nu,\varepsilon}$.

The variation of constants formulas provide leading-order control over $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the intervals $I_{f,\varepsilon}^\pm$ and $I_{s,\varepsilon}$. Consequently, we obtain approximations for $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi_{\mathrm{p},\varepsilon}'$ for each $\nu \in \mathcal{S}_\delta$. Substituting these into (5.2) yields the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

This section is structured as follows. First, we establish the aforementioned reduced eigenvalue problems along the pulse (i.e. for $x \in I_{f,\varepsilon}$) and along the slow manifold (i.e. for $x \in I_{s,\varepsilon}$) and we generate exponential trichotomies for these problems. Then, we construct solutions to (3.2) on $I_{s,\varepsilon}$ and $I_{f,\varepsilon}^\pm$ using variation of constants formulas. By matching these solutions at the endpoints of the intervals $I_{f,\varepsilon}^\pm$ and $I_{s,\varepsilon}$ we obtain the desired piecewise continuous solution $\varphi_{\nu,\varepsilon}$ to (3.2) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$. We show that there is a unique $\lambda$-value for which the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes. Finally, we substitute leading-order approximations of $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi_{\mathrm{p},\varepsilon}'$ into the key identity (5.2) and obtain the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

## 5.2 A reduced eigenvalue problem along the pulse

We establish a reduced eigenvalue problem along the pulse by setting $\varepsilon$ and $\lambda$ to 0 in (3.2), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x,u_0)$. Thus, the reduced eigenvalue problem reads

$$\varphi_x = \mathcal{A}_0(x)\varphi, \quad \varphi = (u,p,v,q) \in \mathbb{C}^{2(m+n)}, \tag{5.7}$$

with

$$\mathcal{A}_0(x) := \left(\begin{array}{c|c} \mathcal{A}_1(x) & \mathcal{A}_2(x) \\ \hline \mathcal{A}_3(x) & \mathcal{A}_f(x) \end{array}\right) := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \partial_u H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 & \partial_v H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 \\ \hline 0 & 0 & 0 & D_2^{-1} \\ \partial_u G(u_0, v_{\mathrm{h}}(x,u_0),0) & 0 & \partial_v G(u_0, v_{\mathrm{h}}(x,u_0),0) & 0 \end{array}\right).$$

Note that (5.7) coincides with the variational equation about the pulse solution $\phi_{\mathrm{h}}(x,u_0)$ to the fast reduced system (2.4).

The $u$-components of any solution to (5.7) are constant, whereas the $p$-components are slaved to the other components. Moreover, given the values of the $u$-components, the dynamics in the $v$- and $q$-components is determined by system (3.6) via the variation of constants formula. Therefore, the reduced eigenvalue problem (5.7) is governed by the variational equation (3.6) about the homoclinic $\psi_{\mathrm{h}}(x,u_0)$ to (2.5) at $u = u_0$.

Thus, before studying problem (5.7), we study the dynamics of the fast variational equation (3.6). Naturally, the derivative $\partial_x\psi_{\mathrm{h}}(x,u_0)$ is a non-trivial, exponentially localized solution to (3.6). Moreover, since $\psi_{\mathrm{h}}(0,u_0)$ is contained in the space $\ker(I - R_f)$ by **(E1)**, system (3.6) is $R_f$-reversible at $x = 0$. We establish exponential dichotomies for (3.6) on both half-lines that respect the reversible symmetry.

**Proposition 5.1.** *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Then, the fast variational equation* (3.6) *admits exponential dichotomies on* $[0,\infty)$ *and* $(-\infty,0]$ *with constants* $C,\mu_r > 0$ *and rank* $n$ *projections* $P_{f,\pm}(x)$ *satisfying*

$$\|P_{f,\pm}(\pm x) - \mathcal{P}_f\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x}, \quad x \geq 0, \tag{5.8}$$

*where* $\mu_{\mathrm{h}} > 0$ *is as in* (2.8), $\mu_r > 0$ *is as in Lemma 3.9 and* $\mathcal{P}_f$ *denotes the spectral projection onto the stable eigenspace of the asymptotic matrix*

$$\mathcal{A}_{f,\infty} := \lim_{x\to\pm\infty} \mathcal{A}_f(x) = \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0,0,0) & 0 \end{pmatrix}. \tag{5.9}$$

*The space of exponentially localized solutions to* (3.6) *is spanned by* $\kappa_{\mathrm{h}}(x) = (\partial_x v_{\mathrm{h}}(x,u_0), \partial_x q_{\mathrm{h}}(x,u_0)) = \partial_x\psi_{\mathrm{h}}(x,u_0)$. *Similarly, the adjoint* (3.7) *has a non-trivial, exponentially localized solution* $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$, *which is unique up to scalar multiples and satisfies*

$$\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x,u_0)dx \neq 0, \quad \|\psi_{\mathrm{ad}}(y)\| \leq Ce^{-\mu_r|y|}, \quad y \in \mathbb{R}.$$

*Moreover, we have the decomposition*

$$\mathbb{C}^{2n} = Y^u \oplus Y^s \oplus Y^c \oplus Y^\perp, \tag{5.10}$$

*with* $Y^c = \mathrm{Sp}(\kappa_{\mathrm{h}}(0))$, $Y^\perp = \mathrm{Sp}(\psi_{\mathrm{ad}}(0))$ *and*

$$\begin{aligned} P_{f,+}(0)[\mathbb{C}^{2n}] &= Y^s \oplus Y^c, & P_{f,-}(0)[\mathbb{C}^{2n}] &= Y^s \oplus Y^\perp, \\ \ker(P_{f,+}(0)) &= Y^u \oplus Y^\perp, & \ker(P_{f,-}(0)) &= Y^u \oplus Y^c. \end{aligned} \tag{5.11}$$

*The spaces* $Y^u \oplus Y^s$, $Y^\perp$ *and* $Y^c$ *are pairwise orthogonal and the decomposition* (5.10) *respects the reversible symmetry:*

$$R_f\kappa_{\mathrm{h}}(0) = -\kappa_{\mathrm{h}}(0), \quad R_f\psi_{\mathrm{ad}}(0) = \psi_{\mathrm{ad}}(0), \quad R_f[Y^s] = Y^u. \tag{5.12}$$

**Proof.** By Lemma 3.9 the asymptotic matrix $\mathcal{A}_{f,\infty}$ is hyperbolic with spectral gap larger than $\mu_r$. The stable and unstable eigenspaces of $\mathcal{A}_{f,\infty}$ have dimension $n$. Moreover, estimate (2.8) implies

$$\left\|\mathcal{A}_f(x) - \mathcal{A}_{f,\infty}\right\| \leq K e^{-\mu_{\mathrm{h}}|x|}, \quad x \in \mathbb{R},$$

for some $K > 0$. Hence, [35, Lemma 3.4] provides exponential dichotomies for (3.6) on $[0,\infty)$ and $(-\infty, 0]$ with constants $C_r, \mu_r > 0$ and rank $n$ projections $P_{f,\pm}(x)$ satisfying (5.8). In addition, by Proposition 3.10, the kernel of the operator $\mathcal{L}_f$, associated with system (3.6), is one-dimensional, because 0 is a simple root of $\mathcal{E}_{f,0}$. Since $\kappa_{\mathrm{h}}(x)$ is a non-trivial, exponentially localized solution to (3.6) by **(E1)**, we deduce $Y^c := \mathrm{Sp}(\kappa_{\mathrm{h}}(0)) = P_{f,+}(0)[\mathbb{C}^{2n}] \cap \ker(P_{f,-}(0))$.

Define $Y^s$ to be the $(n-1)$-dimensional orthogonal complement of $Y^c$ in $P_{f,+}(0)[\mathbb{C}^{2n}]$. Any solution $\varphi(x)$ to (3.6) with initial condition $\varphi(0) \in Y^s$ decays exponentially to 0 as $x \to \infty$. In addition, since system (3.6) is $R_f$-reversible at $x = 0$, the solution $R_f\varphi(-x)$ to (3.6) decays exponentially to 0 as $x \to -\infty$. Therefore, $Y^u := R_f[Y^s]$ is contained in $\ker(P_{f,-}(0))$. Since $R_f$ is self-adjoint and $R_f[\kappa_{\mathrm{h}}(0)] = -\kappa_{\mathrm{h}}(0)$, the $n$-dimensional space $\ker(P_{f,-}(0))$ arises as the orthogonal sum of $Y^c$ and $Y^u$.

By Remark 3.11 there exists a non-trivial, exponentially localized solution $\psi_{\mathrm{ad}}(x)$ to the adjoint equation (3.7), which is unique up to scalar multiples. Clearly, system (3.7) also has an exponential dichotomy on both half-lines with constants $C_r, \mu_r > 0$. This yields $\|\psi_{\mathrm{ad}}(x)\| \leq C_r e^{-\mu_r|x|}$ for $x \in \mathbb{R}$. The pointwise inner product of $\psi_{\mathrm{ad}}(x)$ with any solution $\varphi(x)$ to (3.6) is constant in $x$. Thus, the pointwise inner product of $\psi_{\mathrm{ad}}(x)$ with solutions $\varphi(x)$ to (3.6) that are decaying to 0 as $x \to \pm\infty$ must equal 0. Hence, the spaces $Y^s \oplus Y^u$, $Y^c$ and $Y^\perp := \mathrm{Sp}(\psi_{\mathrm{ad}}(0))$ must be pairwise orthogonal. Since we have the decomposition (5.10), we may without loss of generality assume by [40, Lemma 1.2(ii)] that $P_{f,-}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^\perp$ and $\ker(P_{f,+}(0)) = Y^u \oplus Y^\perp$.

Finally, $R_f\psi_{\mathrm{ad}}(-x)$ is also an exponentially localized solution to (3.7). This implies $R_f\psi_{\mathrm{ad}}(0) = \alpha\psi_{\mathrm{ad}}(0)$ for some $\alpha \in \sigma(R_f) = \{\pm 1\}$. On the other hand, because the eigenvalue 0 of $\mathcal{L}_f$ has algebraic multiplicity 1 by Proposition 3.10, the generalized eigenvalue problem

$$\mathcal{L}_f\varphi = \partial_x v_{\mathrm{h}}(x, u_0),$$

has no bounded solutions. Hence, the Fredholm alternative in [35, Lemma 4.2] implies

$$0 \neq \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x, u_0) dx,$$

since we have $\ker(\mathcal{L}_f^*) = \mathrm{Sp}(\psi_{\mathrm{ad},2})$. Therefore, $\psi_{\mathrm{ad},2}(x)$ cannot be even, because $\partial_x v_{\mathrm{h}}(x, u_0)$ is an odd function of $x$. Hence, $\psi_{\mathrm{ad},2}(x)$ is odd and we establish $R_f\psi_{\mathrm{ad}}(0) = \psi_{\mathrm{ad}}(0)$. □

The evolution operator of the reduced eigenvalue problem (5.7) can be expressed in terms of the evolution operator of the fast variational equation (3.6) via the variation of constants formula. Thus, the solution $\kappa_{\mathrm{h}}(x) = \partial_x\psi_{\mathrm{h}}(x, u_0)$ to (3.6) yields the non-trivial, exponentially localized solution

$$\varphi_{\mathrm{h}}(x) := \begin{pmatrix} \int_\infty^x \mathcal{A}_2(z)\kappa_{\mathrm{h}}(z)dz \\ \kappa_{\mathrm{h}}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ H_2(u_0, v_{\mathrm{h}}(x, u_0)) \\ \partial_x v_{\mathrm{h}}(x, u_0) \\ \partial_x q_{\mathrm{h}}(x, u_0) \end{pmatrix} = \partial_x\phi_{\mathrm{h}}(x, u_0), \tag{5.13}$$

to (5.7). Moreover, since the matrix function $\mathcal{K}_{in}(x) := (\partial_u\psi_{\mathrm{h}}(x, u_0) \mid 0)$ solves the inhomogeneous problem

$$\mathcal{X}_x = \mathcal{A}_f(x)\mathcal{X} + \mathcal{A}_3(x), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}),$$

we obtain a family of solutions

$$\Phi_{in}(x) := \begin{pmatrix} I + \int_0^x [\mathcal{A}_2(z)\mathcal{K}_{in}(z) + \mathcal{A}_1(z)]\, dz \\ \mathcal{K}_{in}(x) \end{pmatrix}. \tag{5.14}$$

to (5.7). By **(S1)** and (2.8) there exists a constant $C > 0$ such that

$$\left\| \Phi_{in}(\pm x) - \begin{pmatrix} \Upsilon_{\pm\infty} \\ 0 \end{pmatrix} \right\| \leq Ce^{-\mu_{\mathrm{h}} x}, \quad x \geq 0, \tag{5.15}$$

with

$$\Upsilon_{\pm\infty} := \begin{pmatrix} I & 0 \\ \pm\partial_u \mathcal{J}(u_0) & I \end{pmatrix} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C}),$$

where $\mathcal{J}\colon U_{\mathrm{h}} \to \mathbb{R}$ is defined in (2.7).

We show that the exponential dichotomies of (3.6), established in Proposition 5.1, yield exponential trichotomies for (5.7) with projections converging to the spectral projections of the associated asymptotic matrix

$$\mathcal{A}_\infty := \lim_{x\to\pm\infty} \mathcal{A}_0(x) = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty} \\ 0 & \mathcal{A}_{f,\infty} \end{pmatrix}, \tag{5.16}$$

where $\mathcal{A}_{f,\infty}$ is defined in (5.9) and

$$\mathcal{A}_{2,\infty} := \begin{pmatrix} 0 & 0 \\ \partial_v H_2(u_0, 0, 0) & 0 \end{pmatrix}. \tag{5.17}$$

**Proposition 5.2.** *Suppose* $0$ *is a simple root of* $\mathcal{E}_{f,0}$. *System* (5.7) *admits exponential trichotomies on* $[0,\infty)$ *and* $(-\infty, 0]$ *with constants* $C, \mu_r > 0$ *and projections* $P_\pm^{u,s,c}(x)$ *satisfying*

$$\left\| P_\pm^{u,s,c}(\pm x) - \mathcal{P}^{u,s,c} \right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x/2}, \quad x \geq 0, \tag{5.18}$$

*where* $\mu_{\mathrm{h}} > 0$ *is as in* (2.8), $\mu_r > 0$ *is as in Lemma 3.9 and* $\mathcal{P}^u, \mathcal{P}^s$ *and* $\mathcal{P}^c$ *are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix* $\mathcal{A}_\infty$*, respectively. Moreover, it holds*

$$\begin{aligned} P_-^u(0) &= \begin{pmatrix} 0 & \int_{-\infty}^0 \mathcal{A}_2(x)\Phi_{f,-}^u(x,0)dx \\ 0 & I - P_{f,-}(0) \end{pmatrix}, & P_+^u(0) &= \begin{pmatrix} 0 & 0 \\ \int_0^\infty \Phi_{f,+}^u(0,x)\mathcal{A}_3(x)dx & I - P_{f,+}(0) \end{pmatrix}, \\ P_+^s(0) &= \begin{pmatrix} 0 & \int_\infty^0 \mathcal{A}_2(x)\Phi_{f,+}^s(x,0)dx \\ 0 & P_{f,+}(0) \end{pmatrix}, & P_-^s(0) &= \begin{pmatrix} 0 & 0 \\ \int_0^{-\infty} \Phi_{f,-}^s(0,x)\mathcal{A}_3(x)dx & P_{f,-}(0) \end{pmatrix}, \end{aligned} \tag{5.19}$$

*where* $\Phi_{f,\pm}^{u,s}(x,y)$ *denotes the (un)stable evolution operator of the fast variational equation* (3.6) *under the exponential dichotomies, established in Proposition 5.1, with projections* $P_{f,\pm}(x)$*. Finally, we have the decompositions*

$$\ker(P_+^u(0)) = P_+^s(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \quad \ker(P_-^s(0)) = P_-^u(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \tag{5.20}$$

*where* $\Phi_{in}$ *is defined in* (5.14)*, and*

$$\begin{aligned} P_+^s(0)[\mathbb{C}^{2(m+n)}] &= P_+^s(0)[Z^s] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad & Z^s &:= \{(0,b)\colon b \in Y^s\}, \\ P_-^u(0)[\mathbb{C}^{2(m+n)}] &= P_-^u(0)[Z^u] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad & Z^u &:= \{(0,b)\colon b \in Y^u\}. \end{aligned} \tag{5.21}$$

*where* $Y^{u,s}$ *are as in Proposition 5.1 and* $\varphi_{\mathrm{h}}$ *is defined in* (5.13).

**Proof.** In the following, we denote by $C > 0$ a sufficiently large constant.

The evolution $\Phi_0(x,y)$ of (5.7) can be expressed in terms of the evolution $\Phi_f(x,y)$ of (3.6) as follows

$$\Phi_0(x,y) = \begin{pmatrix} I + \int_y^x \left[\mathcal{A}_2(z) \int_y^z \Phi_f(z,w)\mathcal{A}_3(w)dw + \mathcal{A}_1(z)\right] dz & \int_y^x \mathcal{A}_2(z)\Phi_f(z,y)dz \\ \int_y^x \Phi_f(x,z)\mathcal{A}_3(z)dz & \Phi_f(x,y) \end{pmatrix}. \tag{5.22}$$

By Proposition 5.1 equation (3.6) admits exponential dichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C, \mu_r > 0$ and rank $n$ projections $P_{f,\pm}(x)$ satisfying

$$\left\|P_{f,\pm}(\pm x) - \mathcal{P}_f\right\| \leq Ce^{-\min\{\mu_r,\mu_\mathrm{h}\}x}, \quad x \geq 0, \tag{5.23}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$, defined in (5.9). We construct an explicit exponential trichotomy for (5.7) on $(-\infty,0]$ using the matrix functions

$$A(x) := \int_{-\infty}^x \mathcal{A}_2(z)\Phi^u_{f,-}(z,x)dx, \qquad B(x) := \int_x^0 \Phi^u_{f,-}(x,z)\mathcal{A}_3(z)dz,$$
$$E(x) := \int_0^x \mathcal{A}_2(z)\Phi^s_{f,-}(z,x)dx, \qquad D(x) := \int_x^{-\infty} \Phi^s_{f,-}(x,z)\mathcal{A}_3(z)dz.$$

Clearly, $A, B, D$ and $E$ are bounded on $(-\infty,0]$. We consider their asymptotic behavior. By (2.8) and **(S1)**, it holds

$$\|\mathcal{A}_1(x)\|, \|\mathcal{A}_2(x) - \mathcal{A}_{2,\infty}\|, \|\mathcal{A}_3(x)\|, \|\mathcal{A}_f(x) - \mathcal{A}_{f,\infty}\| \leq Ce^{-\mu_\mathrm{h}|x|}. \quad x \in \mathbb{R}, \tag{5.24}$$

By writing $B(x)$ as a sum of two integrals over the intervals $(x, x/2)$ and $(x/2, 0)$ and estimating both integrals independently using (5.24) and the exponential dichotomy of (3.6), we deduce that $B(x)$ converges exponentially to 0 as $x \to -\infty$ with rate $\min\{\mu_r,\mu_\mathrm{h}\}/2$. Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 3.9, the matrix $\mathcal{A}_f(x)$ is by (5.24) invertible for $x < 0$ sufficiently small. Thus, for $x \ll 0$ we may write

$$A(x) = \int_{-\infty}^x \mathcal{A}_2(z)\mathcal{A}_f(z)^{-1}\partial_z\Phi^u_{f,-}(z,x)dz.$$

Combining the latter with (5.23) and (5.24), leads, via integration by parts, to the approximations

$$\|B(x)\|, \left\|A(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I - \mathcal{P}_f)\right\| \leq Ce^{\min\{\mu_r,\mu_\mathrm{h}\}x/2}, \quad x \leq 0. \tag{5.25}$$

Similarly, we derive

$$\|D(x)\|, \left\|E(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f\right\| \leq Ce^{\min\{\mu_r,\mu_\mathrm{h}\}x/2}, \quad x \leq 0. \tag{5.26}$$

We define candidate trichotomy projections

$$P^u_-(x) := \begin{pmatrix} A(x)B(x) & A(x) \\ B(x) & I - P_{f,-}(x) \end{pmatrix}, \quad P^s_-(x) := \begin{pmatrix} E(x)D(x) & E(x) \\ D(x) & P_{f,-}(x) \end{pmatrix}, \quad x \leq 0,$$

and we calculate using (5.22)

$$\begin{aligned} P^u_-(x)\Phi_0(x,y) &= \begin{pmatrix} A(x)\Phi^u_{f,-}(x,y)B(y) & A(x)\Phi^u_{f,-}(x,y) \\ \Phi^u_{f,-}(x,y)B(y) & \Phi^u_{f,-}(x,y) \end{pmatrix} = \Phi_0(x,y)P^u_-(y), \\ P^s_-(y)\Phi_0(y,x) &= \begin{pmatrix} E(y)\Phi^s_{f,-}(y,x)D(x) & E(y)\Phi^s_{f,-}(y,x) \\ \Phi^s_{f,-}(y,x)D(x) & \Phi^s_{f,-}(y,x) \end{pmatrix} = \Phi_0(y,x)P^s_-(x), \end{aligned} \qquad x \leq y \leq 0.$$

Since $A, B, D$ and $E$ are bounded on $(-\infty,0]$, the above calculations imply

$$\left\|P^u_-(x)\Phi_0(x,y)\right\|, \left\|P^s_-(y)\Phi_0(y,x)\right\| \leq Ce^{-\mu_r(y-x)}, \quad x \leq y \leq 0.$$

Define $P_-^c(x) := I - P_-^s(x) - P_-^u(x)$ for $x \leq 0$. Observe that

$$P_-^c(x)\Phi_0(x,y) = \begin{pmatrix} E_1(x,y) & E_2(x,y) \\ E_3(x,y) & 0 \end{pmatrix} = \Phi_0(x,y)P_-^c(y), \quad x,y \leq 0,$$

where the matrices

$$\begin{aligned}
E_1(x,y) &:= I + \int_y^x \mathcal{A}_1(z)dz + \int_y^{-\infty} \mathcal{A}_2(z)\int_y^z \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz \\
&\quad + \int_x^{-\infty} \mathcal{A}_2(z)\int_z^0 \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz + \int_y^0 \mathcal{A}_2(z)\int_y^z \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz \\
&\quad + \int_x^0 \mathcal{A}_2(z)\int_z^{-\infty} \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz, \\
E_2(x,y) &:= \int_y^{-\infty} \mathcal{A}_2(z)\Phi_{f,-}^u(z,y)dz + \int_y^0 \mathcal{A}_2(z)\Phi_{f,-}^s(z,y)dz, \\
E_3(x,y) &:= \int_0^x \Phi_{f,-}^u(x,z)\mathcal{A}_3(z)dz + \int_{-\infty}^x \Phi_{f,-}^s(x,z)\mathcal{A}_3(z)dz.
\end{aligned}$$

are bounded on $(-\infty,0] \times (-\infty,0]$ by (5.24). Therefore, the projections $P_-^{u,s,c}(x)$ define an exponential trichotomy for equation (5.7) on $(-\infty,0]$. The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by

$$\mathcal{P}^u = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I-\mathcal{P}_f) \\ 0 & I-\mathcal{P}_f \end{pmatrix}, \quad \mathcal{P}^s = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f \\ 0 & \mathcal{P}_f \end{pmatrix}, \quad \mathcal{P}^c = \begin{pmatrix} I & -\mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1} \\ 0 & 0 \end{pmatrix}, \tag{5.27}$$

Thus, the approximations (5.23), (5.25) and (5.26) yield $\|P_-^{u,s,c}(x) - \mathcal{P}^{u,s,c}\| \leq Ce^{\min\{\mu_r,\mu_h\}x/2}$ for $x \leq 0$. We have obtained the desired exponential trichotomy for (5.7) on $(-\infty,0]$. The construction of the exponential trichotomy for (5.7) on $[0,\infty)$ is analogous.

Finally, we establish the decompositions (5.20) and (5.21). On the one hand, the upper $(2m \times 2m)$-block of $\Phi_{in}(0)$ is lower-triangular and has determinant 1. Therefore, the columns of $\Phi_{in}(x)$ constitute $2m$ linearly independent solutions to (5.7), which are bounded, but not exponentially localized by (5.15). On the other hand, $P_\pm^{u,s}(0)$ has rank $n$, since $P_{f,\pm}(0)$ is a rank $n$ projection. This yields the decomposition (5.20). Furthermore, it holds $P_+^s(0)[\mathbb{C}^{2(m+n)}] = P_+^s(0)[\{(0,b) : b \in P_{f,+}(0)[\mathbb{C}^{2n}]\}]$. Since we have $P_{f,+}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^c$ with $Y^c = \mathrm{Sp}(\kappa_h(0))$ by Proposition 5.1, the decomposition of $P_+^s(0)[\mathbb{C}^{2(m+n)}]$ in (5.21) follows. Analogously, we obtain the decomposition of $P_-^u(0)[\mathbb{C}^{2(m+n)}]$ in (5.21). □

As mentioned in §5.1, our goal is to construct a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.2), which has a jump at $x = 0$ only. The solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ arises by matching solutions to (3.2), which are defined on the three intervals $I_{s,\varepsilon}$, $I_{f,\varepsilon}^-$ and $I_{f,\varepsilon}^+$, given by (5.4) and (5.5). We match these solutions in such a way that the jump at 0 is confined to the one-dimensional space spanned by $(0,\psi_{\mathrm{ad}}(0))$, where $\psi_{\mathrm{ad}}(x)$ is the exponentially localized solution to the adjoint variational equation (3.6), established in Proposition 5.1. This requires the following technical result.

**Lemma 5.3.** *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Let* $Y^s, Y^u, Y^c$ *and* $Y^\perp$ *be as in Proposition 5.1. Denote by* $Q^c$ *the projection on* $Y^c$ *along* $Y^s \oplus Y^u \oplus Y^\perp$, *by* $Q^s$ *the projection on* $Y^s$ *along* $Y^u \oplus Y^c \oplus Y^\perp$ *and by* $Q^u$ *the projection on* $Y^u$ *along* $Y^s \oplus Y^c \oplus Y^\perp$. *The projections*

$$\begin{aligned}
\mathcal{Q}^c &:= \begin{pmatrix} 0 & 0 \\ 0 & Q^c \end{pmatrix}, \quad \hat{\mathcal{Q}}^c := \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dxQ^u - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^s+Q^c) \\ 0 & 0 \end{pmatrix}, \\
\mathcal{Q}^s &:= \begin{pmatrix} 0 & 0 \\ Q^s\int_0^{-\infty}\Phi_f(0,x)\mathcal{A}_3(x)dx & Q^s \end{pmatrix}, \quad \mathcal{Q}^u := \begin{pmatrix} 0 & 0 \\ Q^u\int_0^{\infty}\Phi_f(0,x)\mathcal{A}_3(x)dx & Q^u \end{pmatrix},
\end{aligned} \tag{5.28}$$

*are well-defined and it holds*

$$Z^\perp = \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u), \quad Z^\perp := \{(0,b) \colon b \in Y^\perp\}. \tag{5.29}$$

*Moreover, we have*

$$\begin{aligned} &Q^c \Phi_{in}(0) = 0, \quad Q^c \varphi_{\mathrm{h}}(0) = \begin{pmatrix} 0 \\ \kappa_{\mathrm{h}}(0) \end{pmatrix}, \quad \hat{Q}^c \begin{pmatrix} 0 & 0 \\ 0 & I + R_f \end{pmatrix} = 0, \quad \hat{Q}^c \Phi_{in}(0) = \begin{pmatrix} I \\ 0 \end{pmatrix}, \\ &\hat{Q}^c = \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^u + Q^c) - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dxQ^s \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{5.30}$$

*where $\varphi_{\mathrm{h}}$ and $\Phi_{in}$ are defined in* (5.13) *and* (5.14)*, respectively, and $\kappa_{\mathrm{h}}(x) = \partial_x \psi_{\mathrm{h}}(x, u_0)$.*

**Proof.** The integrals in (5.28) converge by (5.11). Thus, the projections in (5.28) are well-defined. Furthermore, the homoclinic solution $\psi_{\mathrm{h}}(x, u_0)$ to (2.5) at $u = u_0$ satisfies $R_f \psi_{\mathrm{h}}(x, u_0) = \psi_{\mathrm{h}}(-x, u_0)$ for any $x \in \mathbb{R}$ by **(E1)**. Taking derivatives yields

$$R_f \kappa_{\mathrm{h}}(0) = -\kappa_{\mathrm{h}}(0), \quad R_f \kappa_{in}(0) = \kappa_{in}(0), \tag{5.31}$$

where $\kappa_{in}(x) = \partial_u \psi_{\mathrm{h}}(x, u_0)$. Consequently, any column of $\kappa_{in}(0)$ lies in the orthogonal complement of $Y^c = \mathrm{Sp}(\kappa_{\mathrm{h}}(0))$, which is given by $Y^s \oplus Y^u \oplus Y^\perp$ by Proposition 5.1. Hence, we have $Q^c \kappa_{in}(0) = 0$ and the first two identities in (5.30) follow.

The fast variational equation (3.6) is $R_f$-reversible at $x = 0$ by **(E1)**. Thus, by (5.12) it holds $\Phi_f(-x, 0)Q^u = R_f \Phi_f(x, 0)Q^s R_f$ and $\Phi_f(-x, 0)Q^c = R_f \Phi_f(x, 0)Q^c R_f$ for any $x \geq 0$. Combining the latter with (5.31) leads to the other three identities in (5.30), where we use that $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds for any $x \in \mathbb{R}$ by **(E1)**.

Using (5.11) we immediately establish $Z^\perp \subset \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$. Conversely, assume $(a, b) \in \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$ with $a \in \mathbb{C}^{2m}$ and $b \in \mathbb{C}^{2n}$. Then, it holds

$$\begin{aligned} &Q^c b = 0, \quad a = \int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dxQ^u b + \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^s + Q^c)b, \\ &Q^s b = -Q^s \int_0^{-\infty} \Phi_f(0,x)\mathcal{A}_3(x)adx, \quad Q^u b = -Q^u \int_0^\infty \Phi_f(0,x)\mathcal{A}_3(x)adx. \end{aligned}$$

We derive that $\mathcal{A}_3(x)a = 0$ for any $x \in \mathbb{R}$. Hence, it holds $Q^{u,s,c}b = 0$ yielding $b \in Y^\perp$ and $a = 0$. We conclude $(a, b) \in Z^\perp$. □

### 5.3 A reduced eigenvalue problem along the slow manifold

Along the slow manifold, i.e. away from the pulse, the $v$-components of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x)$ are exponentially small and the $u$-components are approximated by $u_{\mathrm{s}}(\varepsilon x)$ – see Theorem 2.2. Hence, by assumption **(S1)**, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in the full eigenvalue problem (3.2) is exponentially small, whereas the upper-left block $\mathcal{A}_{11,\varepsilon}(x, \lambda)$ is approximated by $\varepsilon \mathcal{A}_s(\varepsilon x)$, where $\mathcal{A}_s$ is the coefficient matrix of the slow variational equation (2.9). Thus, along the slow manifold, we arrive at the reduced eigenvalue problem

$$\varphi_x = \mathcal{A}_{*,\varepsilon}(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{5.32}$$

with

$$\mathcal{A}_{*,\varepsilon}(x, \lambda) := \begin{pmatrix} \varepsilon \mathcal{A}_s(\varepsilon x) & \mathcal{A}_{12,\varepsilon}(x) \\ 0 & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix}.$$

Due to its upper-triangular block structure, the dynamics in system (5.32) is governed by the blocks on the diagonal via the variation of constants formula. The lower-right block $\mathcal{A}_{22,\varepsilon}(x,\lambda)$ has slowly varying coefficients and is pointwise hyperbolic along the slow manifold. Hence, on the interval $I_{s,\varepsilon}$, defined in (5.4), system

$$\psi_x = \mathcal{A}_{22,\varepsilon}(x,\lambda)\psi, \quad \psi \in \mathbb{C}^{2n}, \tag{5.33}$$

admits an exponential dichotomy, which yields an exponential trichotomy for the reduced eigenvalue problem (5.32).

**Proposition 5.4.** *Provided $\varsigma, \varepsilon > 0$ are sufficiently small, system* (5.32) *has for every $\lambda \in B(0,\varsigma)$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of $\varepsilon$ and $\lambda$, and projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$. Moreover, we have $\mu_s = \frac{1}{2}\mu_r$, where $\mu_r > 0$ is as in Lemma 3.9. The projections $P^{u,s,c}_{*,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in I_{s,\varepsilon}$ and satisfy*

$$\left\|P^{u,s,c}_{*,\varepsilon}(\Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\|, \left\|P^{u,s,c}_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.34}$$

*where $\Xi_\varepsilon$ is as in* (5.4) *and $\mathcal{P}^u, \mathcal{P}^s$ and $\mathcal{P}^c$ are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$, defined in* (5.16)*.*

**Proof.** In the following, we denote by $C > 0$ a constant, which is independent of $\varepsilon$ and $\lambda$.

We start by establishing an exponential dichotomy for the subsystem (5.33) of the reduced eigenvalue problem (5.32). We define

$$J_{\alpha,\varepsilon} := [\Xi_\varepsilon/\alpha, 2L_\varepsilon - \Xi_\varepsilon/\alpha], \quad \alpha \geq 0.$$

First, by Theorem 2.2 it holds

$$\|u'_{\mathrm{p},\varepsilon}(x)\| = \varepsilon\left\|D_1^{-1}p_{\mathrm{p},\varepsilon}(x)\right\| \leq C\varepsilon, \quad \|v'_{\mathrm{p},\varepsilon}(x)\| = \|D_2^{-1}q_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\partial_x\mathcal{A}_{22,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma).$$

Second, by Theorem 2.2 we have

$$\left\|\hat{\phi}_{\mathrm{p},\varepsilon}(x) - (u_{\mathrm{p},\varepsilon}(x),0)\right\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - A(u_{\mathrm{p},\varepsilon}(x),\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.35}$$

where $A(u,\lambda)$ is defined in (3.3). By Theorem 2.2 and Lemma 3.9, the matrix $A(u_{\mathrm{p},\varepsilon}(x),\lambda)$ is, provided $\varepsilon > 0$ is sufficiently small, hyperbolic for each $x \in J_{4,\varepsilon}$ and $\lambda \in \mathcal{B}(0,\varsigma)$ with spectral gap larger than $\mu_r = 2\mu_s$. So, by (5.35), the same holds for $\mathcal{A}_{22,\varepsilon}(x,\lambda)$, provided $\varepsilon > 0$ is sufficiently small. Third, $\mathcal{A}_{22,\varepsilon}$ is bounded on $J_{4,\varepsilon} \times B(0,\varsigma)$ by an $\varepsilon$-independent constant using Theorem 2.2. Combining these three items with Proposition A.3 yields, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy for system (5.33) on $J_{2,\varepsilon}$ with constants $C, \mu_s > 0$ and projections $\Pi_{f,\varepsilon}(x,\lambda)$. The projections $\Pi_{f,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in J_{2,\varepsilon}$ and satisfy

$$\left\|\Pi_{f,\varepsilon}(x,\lambda) - Q_\varepsilon(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{2,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.36}$$

where $Q_\varepsilon(x,\lambda)$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{22,\varepsilon}(x,\lambda)$. On the other hand, by Theorem 2.2 and estimate (2.8) we have

$$\left\|\hat{\phi}_{\mathrm{p},\varepsilon}(\Xi_\varepsilon) - (u_0,0)\right\| \leq C\varepsilon|\log(\varepsilon)|,$$

yielding

$$\left\|\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{A}_{f,\infty}\right\|\leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right),\quad \lambda\in B(0,\varsigma),$$

where $\mathcal{A}_{f,\infty}$ is given by (5.9). Thus, the same bound holds true for the spectral projections associated with $\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)$ and $\mathcal{A}_{f,\infty}$. Combining the latter with (5.36) yields

$$\left\|\Pi_{f,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{P}_f\right\|\leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right),\quad \lambda\in B(0,\varsigma),\tag{5.37}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$.

The next step is to express the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ of the upper-triangular block system (5.32) in terms of the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ of (5.33) and the evolution $\Phi_s(\check{x},\check{y})$ of the slow variational equation (2.9). Thus, via the variation of constants formula we obtain

$$\mathcal{T}_{*,\varepsilon}(x,y,\lambda)=\begin{pmatrix}\Phi_s(\varepsilon x,\varepsilon y) & \int_y^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}_{f,\varepsilon}(z,y,\lambda)dz\\ 0 & \mathcal{T}_{f,\varepsilon}(x,y,\lambda)\end{pmatrix}.\tag{5.38}$$

We define candidate trichotomy projections

$$\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix}0 & \int_{2L_\varepsilon-\frac12\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & \Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix},\\
P^u_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix}0 & \int_{\frac12\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & I-\Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix}, \qquad x\in I_{s,\varepsilon},\ \lambda\in B(0,\varsigma),\\
P^c_{*,\varepsilon}(x,\lambda) &:= I-P^s_{*,\varepsilon}(x,\lambda)-P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}$$

where $\mathcal{T}^{u,s}_{f,\varepsilon}(x,y,\lambda)$ denotes the (un)stable evolution under the exponential dichotomy of (5.33) on $J_{2,\varepsilon}$. The projections $P^{u,s,c}_{*,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x\in I_{s,\varepsilon}$, because the projections $\Pi_{f,\varepsilon}(x,\lambda)$ and the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ are analytic in $\lambda$ using [28, Lemma 2.1.4]. On the other hand, Grönwall's inequality yields

$$\|\Phi_s(\varepsilon x,\varepsilon z)\|\leq C,\quad x,y\in J_{2,\varepsilon},\tag{5.39}$$

because it holds $|\varepsilon L_\varepsilon-\ell_0|\leq C\varepsilon$ by Theorem 2.2. Using (5.38) we calculate for $x,y\in I_{s,\varepsilon}$ and $\lambda\in B(0,\varsigma)$

$$\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) &:= \begin{pmatrix}0 & \int_{2L_\varepsilon-\frac12\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz\\ 0 & \mathcal{T}^s_{f,\varepsilon}(x,y,\lambda)\end{pmatrix}=\mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^s_{*,\varepsilon}(y,\lambda),\\
P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda) &:= \begin{pmatrix}0 & \int_{\frac12\Xi_\varepsilon}^y \Phi_s(\varepsilon y,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & \mathcal{T}^u_{f,\varepsilon}(y,x,\lambda)\end{pmatrix}=\mathcal{T}_{*,\varepsilon}(y,x,\lambda)P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}$$

and

$$\begin{aligned}
P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) &:= \begin{pmatrix}\Phi_s(\varepsilon x,\varepsilon y) & E_\varepsilon(x,y,\lambda)\\ 0 & 0\end{pmatrix}=\mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^c_{*,\varepsilon}(y,\lambda),
\end{aligned}\tag{5.40}$$

$$E_\varepsilon(x,y,\lambda):=-\int_{2L_\varepsilon-\frac12\Xi_\varepsilon}^y \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz-\int_{\frac12\Xi_\varepsilon}^y \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,y,\lambda)dz.$$

Using estimate (5.39) and the fact that $\mathcal{A}_{12,\varepsilon}$ is $\varepsilon$-uniformly bounded on $J_{2,\varepsilon}$ by Theorem 2.2, the above calculations imply for $x,y\in I_{s,\varepsilon}$ and $\lambda\in B(0,\varsigma)$

$$\left\|P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\|,\left\|P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda)\right\|\leq Ce^{-\mu_s(x-y)},\quad x\geq y,$$

and

$$\left\|P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\|\leq C.$$

Therefore, the projections $P^{u,s,c}_{*,\varepsilon}(x)$ define an exponential trichotomy for equation (5.32) on $I_{s,\varepsilon}$.

Finally, we establish the approximations (5.34). Define $\tilde{J}_\varepsilon := \left[\frac{1}{2}\Xi_\varepsilon, \frac{3}{2}\Xi_\varepsilon\right]$. First, by Grönwall type estimates [36, Lemma 1] it holds

$$\|\Phi_s(\varepsilon x, \varepsilon y) - I\| \leq C\varepsilon|\log(\varepsilon)|, \quad x, y \in \tilde{J}_\varepsilon, \tag{5.41}$$

Second, by Theorem 2.2 and estimate (2.8) we have

$$\left\|\hat{\phi}_{p,\varepsilon}(x) - (u_0, 0)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in \tilde{J}_\varepsilon,$$

yielding for $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0, \varsigma)$

$$\left\|\mathcal{A}_{12,\varepsilon}(x) - \mathcal{A}_{2,\infty}\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad \left\|\mathcal{A}_{22,\varepsilon}(x, \lambda) - \mathcal{A}_{f,\infty}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.42}$$

where $\mathcal{A}_{2,\infty}$ is defined in (5.17). Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 3.9, the matrix $\mathcal{A}_{22,\varepsilon}(x, \lambda)$ is by (5.42) invertible for each $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0, \varsigma)$, provided $\varepsilon, \varsigma > 0$ are sufficiently small. Thus, for $\lambda \in B(0, \varsigma)$ we may write

$$\begin{aligned}&\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz \\ &\qquad = \int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{A}_{22,\varepsilon}(z, \lambda)^{-1}\partial_z\mathcal{T}_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz\left(I - \Pi_{f,\varepsilon}(\Xi_\varepsilon, \lambda)\right).\end{aligned}$$

Combining the latter with (5.37), (5.41) and (5.42), leads, via integration by parts, to the approximation

$$\left\|\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}(I - \mathcal{P}_f)\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.43}$$

for $\lambda \in B(0, \varsigma)$. Similarly, we derive

$$\left\|\int_{\frac{3}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}\mathcal{P}_f\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.44}$$

for $\lambda \in B(0, \varsigma)$. On the other hand, (5.39) yields

$$\left\|\int_{2L_\varepsilon - \frac{1}{2}\Xi_\varepsilon}^{\frac{3}{2}\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz\right\| \leq C\varepsilon, \quad \lambda \in B(0, \varsigma). \tag{5.45}$$

The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by (5.27). Thus, the approximations (5.37), (5.43), (5.44) and (5.45) yield $\|P^{u,s,c}_{*,\varepsilon}(\Xi_\varepsilon, \lambda) - \mathcal{P}^{u,s,c}\| \leq C(\varepsilon|\log(\varepsilon)| + |\lambda|)$ for $\lambda \in B(0, \varsigma)$. The other estimate in (5.34) follows analogously. □

## 5.4 Construction of a piecewise continuous eigenfunction

Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and $\Xi_\varepsilon$ be as in (3.10), (5.3) and (5.4), respectively. We establish for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x, \lambda)$ to the full eigenvalue problem (3.2) on the interval $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$. The construction of $\varphi_{\nu,\varepsilon}$ is based on Lin's method [5, 30, 44].

**Theorem 5.5.** *Suppose* $0$ *is a simple root of* $\mathcal{E}_{f,0}$. *Take* $\delta > 0$. *Provided* $\eta, \varepsilon > 0$ *are sufficiently small, there exists for every* $\lambda \in D_{\eta,\varepsilon}$ *and* $\nu \in \mathcal{S}_\delta$ *a piecewise continuous solution* $\varphi_{\nu,\varepsilon}(x, \lambda)$ *to the full eigenvalue problem* (3.2) *on* $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, *which has a jump only at* $x = 0$, *satisfies*

$$\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda),$$

*and enjoys the estimates*

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \varphi_{\rm h}(x)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad x \in [-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \tag{5.46}$$

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{{\rm p},\varepsilon}(x) + \varepsilon\Phi_{in}(x)\mathcal{B}(\nu)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^3 + |\lambda|\right), \quad x \in \left[-\tfrac{\Xi_\varepsilon}{2}, \tfrac{\Xi_\varepsilon}{2}\right], \tag{5.47}$$

*where* $\mathcal{B}(\nu)$, $\varphi_{\rm h}$ *and* $\Phi_{in}$ *are defined in* (3.13), (5.13) *and* (5.14), *respectively, and* $C > 0$ *is a constant independent of* $\varepsilon, \lambda$ *and* $\nu$. *In addition, for any* $\nu \in \mathcal{S}_\delta$ *there exists a unique* $\lambda$*-value* $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$ *for which the jump of* $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ *vanishes.*

**Proof.** In the following, we denote by $C > 0$ a constant, which is independent of $\varepsilon, \lambda$ and $\nu$.

Let $I_{f,\varepsilon} = I_{f,\varepsilon}^+ \cup I_{f,\varepsilon}^-$ and $I_{s,\varepsilon}$ be as in (5.4) and (5.5). Our approach is to regard the full eigenvalue problem (3.2) as a perturbation of the reduced eigenvalue problems (5.7) and (5.32) on the intervals $I_{f,\varepsilon}$ and $I_{s,\varepsilon}$, respectively. Propositions 5.2 and 5.4 yield exponential trichotomies for (5.7) and (5.32). For $\lambda \in D_{\eta,\varepsilon}$, this leads to variation of constants formulas for solutions to (3.2) on the three intervals $I_{s,\varepsilon}$, $I_{f,\varepsilon}^-$ and $I_{f,\varepsilon}^+$. We match these solutions at the endpoints $0, \pm\Xi_\varepsilon$ and $2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}^\pm$ using the estimates (5.18) and (5.34) on the trichotomy projections and identity (5.29). Thus, we obtain for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.2) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$. For each $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ the jump

$$J_{\nu,\varepsilon}(\lambda) := \lim_{x\downarrow 0}\varphi_{\nu,\varepsilon}(x,\lambda) - \lim_{x\uparrow 0}\varphi_{\nu,\varepsilon}(x,\lambda), \tag{5.48}$$

is contained in the one-dimensional space $Z^\perp$, defined in (5.29). Pairing the jump with the solution $\psi_{\rm ad}(x)$ to the adjoint (3.7), established in Proposition 5.1, leads to an (analytic) equation in $\lambda$ and $\nu$, which has a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

The variation of constants formulas provide leading-order expressions for $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the three intervals $I_{s,\varepsilon}$, $I_{f,\varepsilon}^-$ and $I_{f,\varepsilon}^+$, leading to (5.46). Moreover, since the derivative $\phi'_{{\rm p},\varepsilon}(x)$ is a solution to (3.2) at $\lambda = 0$, we can write $\phi'_{{\rm p},\varepsilon}(x)$ in terms of similar variation of constants formulas on $I_{f,\varepsilon}^\pm$ yielding the leading-order approximation (5.47) for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{{\rm p},\varepsilon}(x)$.

Thus, we start by establishing expressions for solutions to the full eigenvalue problem (3.2) along the pulse, i.e. for $x \in I_{f,\varepsilon} = I_{f,\varepsilon}^- \cup I_{f,\varepsilon}^+$. We regard (3.2) as the perturbation

$$\varphi_x = \left(\mathcal{A}_0(x) + \mathcal{B}_{0,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.7). By Theorem 2.2, the perturbation matrix $\mathcal{B}_{0,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_0(x)$ is bounded by

$$\|\mathcal{B}_{0,\varepsilon}(x,\lambda)\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.49}$$

By Proposition 5.2, system (5.7) has exponential trichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C, \mu_r > 0$ and projections $P_\pm^{u,s,c}(x)$ satisfying (5.18). We denote by $\Phi_{0,\pm}^{s,u,c}(x,y)$ the stable, unstable and neutral evolution operator of system (5.7) under the exponential trichotomies. For convenience, we abbreviate $\Phi_{0,\pm}^{sc}(x,y) = (I - P_\pm^u(x))\Phi_0(x,y)$ and $\Phi_{0,\pm}^{uc}(x,y) = (I - P_\pm^s(x))\Phi_0(x,y)$.

We apply the variation of constants formula. Thus, by the decompositions (5.20) and (5.21), any solution $\varphi_{f,\varepsilon}^+(x,\lambda)$ to (3.2) must satisfy the following integral equation on $I_{f,\varepsilon}^+$:

$$\begin{aligned}\varphi_{f,\varepsilon}^+(x,\lambda) = \Phi_{0,+}^u(x,\Xi_\varepsilon)a_+ + \Phi_{in}(x)b_+ + \int_0^x \Phi_{0,+}^s(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^+(y,\lambda)dy \\ + \varphi_{\rm h}(x)c_+ + \Phi_{0,+}^s(x,0)d_+ + \int_{\Xi_\varepsilon}^x \Phi_{0,+}^{uc}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^+(y,\lambda)dy,\end{aligned} \tag{5.50}$$

for some $a_+ \in P_+^u(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_+ \in \mathbb{C}^{2m}$, $c_+ \in \mathbb{C}$ and $d_+ \in Z^s$, where $Z^s$ is defined in (5.21). Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.49) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi_{f,\varepsilon}^+(x,\lambda)$ to (5.50) on $I_{f,\varepsilon}^+$ using the contraction mapping principle. Note that $\varphi_{f,\varepsilon}^+(x,\lambda)$ is linear in $(a_+, b_+, c_+, d_+)$ and satisfies the bound

$$\sup_{x \in I_{f,\varepsilon}^+} \|\varphi_{f,\varepsilon}^+(x,\lambda)\| \leq C\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.51}$$

by estimate (5.49), taking $\eta, \varepsilon > 0$ smaller if necessary. Similarly, any solution $\varphi_{f,\varepsilon}^-(x,\lambda)$ to (3.2) must satisfy the following integral equation on $I_{f,\varepsilon}^-$:

$$\begin{aligned} \varphi_{f,\varepsilon}^-(x,\lambda) = \Phi_{0,-}^s(x,-\Xi_\varepsilon)a_- + \Phi_{in}(x)b_- + \int_0^x \Phi_{0,-}^u(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^-(y,\lambda)dy \\ + \varphi_{\mathrm{h}}(x)c_- + \Phi_{0,-}^u(x,0)d_- + \int_{-\Xi_\varepsilon}^x \Phi_{0,-}^{sc}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^-(y,\lambda)dy, \end{aligned} \tag{5.52}$$

for some $a_- \in P_-^s(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_- \in \mathbb{C}^{2m}$, $c_- \in \mathbb{C}$ and $d_- \in Z^u$, where $Z^u$ is defined in (5.21). There exists for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi_{f,\varepsilon}^-(x,\lambda)$ to (5.52) on $I_{f,\varepsilon}^-$, which is linear in $(a_-, b_-, c_-, d_-)$ and satisfies the bound

$$\sup_{x \in I_{f,\varepsilon}^-} \|\varphi_{f,\varepsilon}^-(x,\lambda)\| \leq C\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.53}$$

taking $\eta, \varepsilon > 0$ smaller if necessary.

Our next step is to obtain expressions for solutions to the full eigenvalue problem (3.2) along the slow manifold, i.e. for $x \in I_{s,\varepsilon}$. We regard (3.2) as the perturbation

$$\varphi_x = \left(\mathcal{A}_{*,\varepsilon}(x,\lambda) + \mathcal{B}_{*,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.32). By Theorem 2.2 it holds

$$\|u_{\mathrm{p},\varepsilon}(x) - u_{\mathrm{s}}(\varepsilon x)\| \leq C\varepsilon, \quad \|v_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in I_{s,\varepsilon}.$$

Therefore, by **(S1)** the perturbation matrix $\mathcal{B}_{*,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_{*,\varepsilon}(x,\lambda)$ is bounded by

$$\left\|\mathcal{B}_{*,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon\left(\varepsilon + |\lambda|\right), \quad x \in I_{s,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.54}$$

By Proposition 5.4 system (5.32) admits for every $\lambda \in D_{\eta,\varepsilon}$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of $\varepsilon$ and $\lambda$, and projections $P_{*,\varepsilon}^{u,s,c}(x,\lambda)$ satisfying (5.34). We denote by $\mathcal{T}_{*,\varepsilon}^{s,u,c}(x,y,\lambda)$ the stable, unstable and neutral evolution operator of system (5.32) under the exponential trichotomy.

We apply the variation of constants formula. Thus, any solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (3.2) must satisfy the following integral equation on $I_{s,\varepsilon}$:

$$\begin{aligned} \varphi_{s,\varepsilon}(x,\lambda) = \mathcal{T}_{*,\varepsilon}^s(x,\Xi_\varepsilon,\lambda)f + \mathcal{T}_{*,\varepsilon}^c(x,\Xi_\varepsilon,\lambda)h + \int_{\Xi_\varepsilon}^x \mathcal{T}_{*,\varepsilon}^{sc}(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy \\ + \mathcal{T}_{*,\varepsilon}^u(x,2L_\varepsilon - \Xi_\varepsilon,\lambda)g + \int_{2L_\varepsilon - \Xi_\varepsilon}^x \mathcal{T}_{*,\varepsilon}^u(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy, \end{aligned} \tag{5.55}$$

for some $f \in P_{*,\varepsilon}^s(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $g \in P_{*,\varepsilon}^u(2L_\varepsilon - \Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$ and $h \in P_{*,\varepsilon}^c(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$. Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.54) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (5.55) on $I_{s,\varepsilon}$ using the contraction mapping principle. The solution $\varphi_{s,\varepsilon}(x,\lambda)$ is linear in $(f,g,h)$ and enjoys the bound

$$\sup_{x \in I_{s,\varepsilon}} \|\varphi_{s,\varepsilon}(x,\lambda)\| \leq C\left(\|f\| + \|g\| + \|h\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.56}$$

using estimate (5.54) and the fact that $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ by Theorem 2.2.

Now, we match the solutions $\varphi_{f,\varepsilon}^{\pm}(x,\lambda)$ and $\varphi_{s,\varepsilon}(x,\lambda)$, given by (5.50), (5.52) and (5.55), at the endpoints $x = \pm\Xi_\varepsilon$ and $x = 2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}^{\pm}$. Applying the projection $P_{*,\varepsilon}^{s}(\Xi_\varepsilon,\lambda)$ to the difference $\varphi_{f,\varepsilon}^{+}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition

$$\begin{aligned} f &= \mathcal{H}_{\varepsilon,\lambda}^{1}(a_+,b_+,c_+,d_+),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{1}(a_+,b_+,c_+,d_+)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.57}$$

where we use (5.18), (5.20), (5.34), (5.49) and (5.51) to obtain the bound on the linear map $\mathcal{H}_{\varepsilon,\lambda}^{1}$. Similarly, applying $P_{*,\varepsilon}^{u}(\Xi_\varepsilon,\lambda)$ to $\varphi_{f,\varepsilon}^{+}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition

$$\begin{aligned} a_+ &= \mathcal{H}_{\varepsilon,\lambda}^{2}(a_+,b_+,c_+,d_+,f,g,h),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{2}(a_+,b_+,c_+,d_+,f,g,h)\| &\leq C\left[\varepsilon\left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right.\\ &\quad \left. + \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right)\right], \end{aligned} \tag{5.58}$$

where we use (5.18), (5.20), (5.34), (5.49), (5.51), (5.54), (5.56) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bound on the linear map $\mathcal{H}_{\varepsilon,\lambda}^{2}$. Finally, applying $P_{*,\varepsilon}^{c}(\Xi_\varepsilon,\lambda)$ to $\varphi_{f,\varepsilon}^{+}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition

$$\begin{aligned} h &= \begin{pmatrix} \Upsilon_\infty b_+ \\ 0 \end{pmatrix} + \mathcal{H}_{\varepsilon,\lambda}^{3}(a_+,b_+,c_+,d_+),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{3}(a_+,b_+,c_+,d_+)\| &\leq \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.59}$$

where we use (5.15), (5.18), (5.34), (5.49) and (5.51) to obtain the bound on the linear map $\mathcal{H}_{\varepsilon,\lambda}^{3}$. Note that $\mathcal{H}_{\varepsilon,\lambda}^{1,2,3}$ are analytic in $\lambda$, because the perturbations matrices $\mathcal{B}_{0,\varepsilon}(x,\lambda)$ and $\mathcal{B}_{*,\varepsilon}(x,\lambda)$, the projections $P_{*,\varepsilon}^{u,s,c}(x,\lambda)$ and the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ are analytic in $\lambda$ by Proposition 5.4 and [28, Lemma 2.1.4].

Take $\nu \in \mathcal{S}_\delta$. We obtain the following matching conditions for any $\lambda \in D_{\eta,\varepsilon}$ by applying the projections $P_{*,\varepsilon}^{u,s,c}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$ to the difference $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - e^{i\nu}\varphi_{f,\varepsilon}^{-}(-\Xi_\varepsilon,\lambda)$:

$$\begin{aligned} g &= \mathcal{H}_{\varepsilon,\lambda}^{4}(a_-,b_-,c_-,d_-),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{4}(a_-,b_-,c_-,d_-)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \end{aligned} \tag{5.60}$$

$$\begin{aligned} a_- &= \mathcal{H}_{\varepsilon,\lambda}^{5}(a_-,b_-,c_-,d_-,f,g,h),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{5}(a_-,b_-,c_-,d_-,f,g,h)\| &\leq C\left[\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right)\right.\\ &\quad \left. + \left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right], \end{aligned} \tag{5.61}$$

$$\begin{aligned} \mathcal{T}_{*,\varepsilon}^{c}(2L_\varepsilon - \Xi_\varepsilon,\Xi_\varepsilon,\lambda)h &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty} b_- \\ 0 \end{pmatrix} + \mathcal{H}_{\varepsilon,\lambda}^{6}(a_-,b_-,c_-,d_-,f,g,h),\\ \|\mathcal{H}_{\varepsilon,\lambda}^{6}(a_-,b_-,c_-,d_-,f,g,h)\| &\leq C\left[\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right)\right.\\ &\quad \left. + \left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right], \end{aligned} \tag{5.62}$$

where we use (5.15), (5.18), (5.20), (5.34), (5.49), (5.53), (5.54), (5.56) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bounds on the linear maps $\mathcal{H}_{\varepsilon,\lambda}^{4,5,6}$, which are analytic in $\lambda$. We introduce the shorthand notation $a = (a_+,a_-)$, $b = (b_+,b_-)$, $c = (c_+,c_-)$ and $d = (d_+,d_-)$. Substituting (5.59) into (5.62) yields a linear map $\mathcal{H}_{\varepsilon,\lambda}^{7}$, which is analytic in $\lambda$, satisfying

$$\begin{aligned} \begin{pmatrix} \Phi_s(2\ell_0,0)\Upsilon_\infty b_+ \\ 0 \end{pmatrix} &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty} b_- \\ 0 \end{pmatrix} + \mathcal{H}_{\varepsilon,\lambda}^{7}(a,b,c,d,f,g,h)\\ \|\mathcal{H}_{\varepsilon,\lambda}^{7}(a,b,c,d,f,g,h)\| &\leq C\left[\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a\| + \|b\| + \|c\| + \|d\|\right)\right.\\ &\quad \left. + \left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right], \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.63}$$

where we use (5.40), $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ and the bound

$$\|\Phi_s(\varepsilon x, \varepsilon y) - I\| \leq C\varepsilon |\log(\varepsilon)|, \quad |x-y| \leq 2\Xi_\varepsilon, \tag{5.64}$$

which follows from Grönwall type estimates [36, Lemma 1]. The matching conditions (5.57), (5.58), (5.59), (5.60), (5.61) and (5.63) constitute a system of 6 linear equations in 11 variables. One readily observes that, provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for $a_\pm, f, g, h$ and $b_-$ yielding linear maps $\mathcal{H}^{8,9}_{\varepsilon,\lambda}$, which are analytic in $\lambda$ and satisfy

$$\begin{aligned}
(f,g,a) &= \mathcal{H}^8_{\varepsilon,\lambda}(b_+, c, d),\\
(h, b_-) &= \left(\begin{pmatrix} \Upsilon_\infty b_+ \\ 0 \end{pmatrix}, e^{-i\nu}\Upsilon_\infty \Phi_s(2\ell_0, 0)\Upsilon_\infty b_+\right) + \mathcal{H}^9_{\varepsilon,\lambda}(b_+, c, d), \qquad \lambda \in D_{\eta,\varepsilon},\\
\|\mathcal{H}^{8,9}_{\varepsilon,\lambda}(b_-, c, d)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|b_-\| + \|c\| + \|d\|\right).
\end{aligned} \tag{5.65}$$

Thus, since the projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ are complementary, we observe that $(f,g,h,a,b_-)$ satisfies (5.65) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon, \lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon, \lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon, \lambda)$ hold true.

Our next step is to match the solutions $\varphi^\pm_{f,\varepsilon}(x,\lambda)$, given by (5.50) and (5.52), at $x = 0$ such that the jump $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ is confined to the one-dimensional space $Z^\perp$, which is defined in (5.29). First, we apply the projections $Q^{u,s}$, given by (5.28). By (5.11) and (5.19) it holds

$$Q^s P^s_-(0) = Q^s, \quad Q^s P^u_+(0) = 0, \quad (I - Q^s P^s_+(0))\,[Z^s] = 0. \tag{5.66}$$

Applying the projection $Q^s$ to the difference $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition

$$\begin{aligned}
d_+ &= \mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d),\\
\|\mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a\| + \|b\| + \|c\| + \|d\|\right),
\end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.67}$$

where we use (5.20), (5.21), (5.49), (5.51), (5.53) and (5.66) to obtain the bound on the linear map $\mathcal{H}^{10}_{\varepsilon,\lambda}$, which is analytic in $\lambda$. Similarly, applying $Q^u$ to $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$, we establish a linear map $\mathcal{H}^{11}_{\varepsilon,\lambda}$, which is analytic in $\lambda$, satisfying

$$\begin{aligned}
d_- &= \mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d),\\
\|\mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a\| + \|b\| + \|c\| + \|d\|\right),
\end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.68}$$

Next, we apply the projections $Q^c$ and $\hat{Q}^c$, given by (5.28). By (5.11) and (5.19) it holds

$$Q^c P^u_-(0) = Q^c = Q^c P^s_+(0), \quad \hat{Q}^c P^{sc}_-(0) = \hat{Q}^c = \hat{Q}^c P^{uc}_+(0). \tag{5.69}$$

Applying $Q^c$ to the difference $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition

$$c_+ = c_-, \tag{5.70}$$

where we use (5.30) and (5.69). Finally, applying $\hat{Q}^c$ to $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition

$$\begin{aligned}
\begin{pmatrix} b_+ - b_- \\ 0 \end{pmatrix} &= \int_{-\Xi_\varepsilon}^0 \hat{Q}^c \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_h(y)c_- dy + \mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d)\\
&\qquad - \int_{\Xi_\varepsilon}^0 \hat{Q}^c \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_h(y)c_+ dy,\\
\|\mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left[\left(\|a\| + \|b\| + \|d\|\right) + |\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\|c\|\right],
\end{aligned} \tag{5.71}$$

where we use (5.30), (5.49), (5.51), (5.53) and (5.69) to obtain the bound on the linear map $\mathcal{H}^{12}_{\varepsilon,\lambda}$, which is analytic in $\lambda$.

We wish to approximate the integral expressions in (5.71). Therefore, we split the perturbation $\mathcal{B}_{0,\varepsilon}(y,\lambda)$ in an $\varepsilon$-dependent and $\lambda$-dependent part, i.e. it holds

$$\left\|\mathcal{B}_{0,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0) - \lambda\mathcal{B}_*\right\| \leq C\varepsilon|\lambda|, \quad y \in I_{f,\varepsilon}, \lambda \in \mathbb{C}, \tag{5.72}$$

with

$$\mathcal{B}_* := \begin{pmatrix} 0 & 0 \\ 0 & B_* \end{pmatrix} \in \mathrm{Mat}_{2(n+m)\times 2(n+m)}(\mathbb{C}), \quad B_* := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}).$$

First, we approximate the $\lambda$-dependent part of the integrals in (5.71). Recall that system (3.6) is $R_f$-reversible at $x = 0$ by **(E1)**. Thus, the evolution $\Phi_f(x,y)$ of (3.6) satisfies $R_f\Phi_f(x,y)R_f = \Phi_f(-x,-y)$ for any $x,y \in \mathbb{R}$. Hence, using (5.22) we calculate

$$\begin{aligned}\Phi_0(0,x)\mathcal{B}_*\varphi_\mathrm{h}(x) &= \begin{pmatrix} \int_x^0 \mathcal{A}_2(y)\Phi_f(y,x)B_*\kappa_\mathrm{h}(x)dy \\ \Phi_f(0,x)B_*\kappa_\mathrm{h}(x) \end{pmatrix} = \begin{pmatrix} -\int_{-x}^0 \mathcal{A}_2(y)\Phi_f(y,-x)B_*\kappa_\mathrm{h}(-x)dy \\ R_f\Phi_f(0,-x)B_*\kappa_\mathrm{h}(-x) \end{pmatrix} \\ &= \begin{pmatrix} -I & 0 \\ 0 & R_f \end{pmatrix}\Phi_0(0,-x)\mathcal{B}_*\varphi_\mathrm{h}(-x),\end{aligned}$$

where we use that $R_fB_* = -B_*R_f$, $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$, $R_f\kappa_\mathrm{h}(x) = -\kappa_\mathrm{h}(-x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds true for any $x \in \mathbb{R}$ by **(E1)**. Combining the latter identity with (5.30) yields

$$\hat{Q}^c\left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_*\varphi_\mathrm{h}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_*\varphi_\mathrm{h}(y)dy\right] = 0. \tag{5.73}$$

Next, we approximate the $\varepsilon$-dependent part of the integrals in (5.71). This can be done by using that the derivative $\phi'_{\mathrm{p},\varepsilon}(x)$ is a solution to (3.2) at $\lambda = 0$. Thus, $\phi'_{\mathrm{p},\varepsilon}(x)$ satisfies the integral equation (5.50) on $I_{f,\varepsilon}^+$ at $\lambda = 0$, i.e. we have for $x \in I_{f,\varepsilon}^+$

$$\begin{aligned}\phi'_{\mathrm{p},\varepsilon}(x) = \Phi_{0,+}^u(x,\Xi_\varepsilon)a_{\mathrm{p},+} + \Phi_{in}(x)b_{\mathrm{p},+} + \int_0^x \Phi_{0,+}^s(x,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_\mathrm{h}(x)c_{\mathrm{p},+} + \Phi_{0,+}^s(x,0)d_{\mathrm{p},+} + \int_{\Xi_\varepsilon}^x \Phi_{0,+}^{uc}(x,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy,\end{aligned} \tag{5.74}$$

for some constants $a_{\mathrm{p},+} \in P_+^u(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},+} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},+} \in \mathbb{C}$ and $d_{\mathrm{p},+} \in Z^s$, where we suppress their $\varepsilon$-dependence for notational convenience. Similarly, it holds for $x \in I_{f,\varepsilon}^-$

$$\begin{aligned}\phi'_{\mathrm{p},\varepsilon}(x) = \Phi_{0,-}^s(x,-\Xi_\varepsilon)a_{\mathrm{p},-} + \Phi_{in}(x)b_{\mathrm{p},-} + \int_0^x \Phi_{0,-}^u(x,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_\mathrm{h}(x)c_{\mathrm{p},-} + \Phi_{0,-}^u(x,0)d_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^x \Phi_{0,-}^{sc}(x,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy,\end{aligned} \tag{5.75}$$

for some $a_{\mathrm{p},-} \in P_-^s(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},-} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},-} \in \mathbb{C}$ and $d_{\mathrm{p},-} \in Z^u$. By applying suitable projections, we obtain leading-order approximations for the constants $a_{\mathrm{p},\pm}, b_{\mathrm{p},\pm}, c_{\mathrm{p},\pm}$ and $d_{\mathrm{p},\pm}$. This leads to the desired approximations for the integrals in (5.71). First, Theorem 2.2 and **(S1)** yield

$$\left\|\phi'_{\mathrm{p},\varepsilon}(\pm\Xi_\varepsilon) - \varepsilon\begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0) \\ 0 \\ 0 \end{pmatrix}\right\| \leq C\varepsilon^2|\log(\varepsilon)|, \tag{5.76}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ solves (2.3). By applying the projections $P_+^u(\Xi_\varepsilon)$ and $P_+^c(\Xi_\varepsilon)$ to (5.74) at $x = \Xi_\varepsilon$, we arrive via (5.20) and (5.21) at

$$a_{\mathrm{p},+} = P_+^u(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon), \quad P_+^c(\Xi_\varepsilon)\Phi_{in}(\Xi_\varepsilon)b_{\mathrm{p},+} = P_+^c(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon).$$

Similarly, we apply $P_-^s(-\Xi_\varepsilon)$ and $P_-^c(-\Xi_\varepsilon)$ to (5.75) at $x = -\Xi_\varepsilon$ yielding

$$a_{\mathrm{p},-} = P_-^s(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon), \quad P_-^c(-\Xi_\varepsilon)\Phi_{in}(-\Xi_\varepsilon)b_{\mathrm{p},-} = P_-^c(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon).$$

Combining the latter two identities with (5.15), (5.18), (5.27) and (5.76) gives

$$\|a_{\mathrm{p},\pm}\| \leq C\varepsilon, \quad \left\| b_{\mathrm{p},\pm} - \varepsilon\Upsilon_{\mp\infty}\begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0) \end{pmatrix}\right\| \leq C\varepsilon^2|\log(\varepsilon)|. \tag{5.77}$$

Recall that we have $\varphi_\mathrm{h}(x) = \partial_x\phi_\mathrm{h}(x,u_0)$. Thus, by Theorem 2.2 it holds

$$\left\|\phi'_{\mathrm{p},\varepsilon}(x) - \varphi_\mathrm{h}(x)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in I_{f,\varepsilon}, \tag{5.78}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ and $\phi_\mathrm{h}$ solve (2.3) and (2.4), respectively. Next, we apply $Q^c$ to (5.74) and (5.75) at $x = 0$, yielding

$$c_{\mathrm{p},+} = \frac{\left\langle \begin{pmatrix} 0 \\ \kappa_\mathrm{h}(0)\end{pmatrix}, \phi'_{\mathrm{p},\varepsilon}(0)\right\rangle}{\|\kappa_\mathrm{h}(0)\|^2} = c_{\mathrm{p},-}, \quad \left|c_{\mathrm{p},\pm} - 1\right| \leq C\varepsilon|\log(\varepsilon)|, \tag{5.79}$$

by (5.30), (5.69) and (5.78). Finally, applying $\hat{Q}^c$ to (5.74) and (5.75) at $x = 0$, gives the identity

$$\begin{aligned}\begin{pmatrix} b_{\mathrm{p},+} - b_{\mathrm{p},-} \\ 0 \end{pmatrix} = \hat{Q}^c\Bigg[&\Phi_{0,-}^s(0,-\Xi_\varepsilon)a_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ &- \Phi_{0,+}^u(0,\Xi_\varepsilon)a_{\mathrm{p},+} - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy\Bigg],\end{aligned}$$

by (5.30) and (5.69). Using (5.49), (5.77) and (5.78), we approximate both sides of the latter identity, yielding

$$\begin{aligned}&\left\|\hat{Q}^c\left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_\mathrm{h}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_\mathrm{h}(y)dy\right] - \varepsilon\begin{pmatrix} 2D_1^{-1}\mathcal{J}(u_0) \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\| \\ &\quad \leq C\varepsilon^2|\log(\varepsilon)|^2,\end{aligned} \tag{5.80}$$

which gives together with (5.72) and (5.73) the desired leading-order expressions of the integrals in (5.71).

Thus, the matching conditions (5.65), (5.67), (5.68), (5.70) and (5.71) constitute a system of 10 linear equations in 11 variables. Provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for $a, b, c_-, d, f, g, h$ yielding linear maps $\mathcal{H}^{13}_{\varepsilon,\lambda}, \mathcal{H}^{14}_{\varepsilon,\lambda}$ and $\mathcal{H}^{15}_{\varepsilon,\lambda}$, which are analytic in $\lambda$, satisfying for $\lambda \in D_{\eta,\varepsilon}$

$$\begin{aligned}(a,d,f,g,h) &= \mathcal{H}^{13}_{\varepsilon,\lambda}(c_+), \\ c_- &= c_+, \\ b_+ &= 2\varepsilon\left(I - e^{-i\nu}\Upsilon_\infty\Phi_\mathrm{s}(2\ell_0,0)\Upsilon_\infty\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix}c_+ + \mathcal{H}^{14}_{\varepsilon,\lambda}(c_+), \\ b_- &= 2\varepsilon\left(e^{i\nu}\Upsilon_{-\infty}\Phi_\mathrm{s}(0,2\ell_0)\Upsilon_{-\infty} - I\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix}c_+ + \mathcal{H}^{15}_{\varepsilon,\lambda}(c_+), \\ \|\mathcal{H}^{13}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)|c_+|, \\ \|\mathcal{H}^{14,15}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2|c_+|,\end{aligned} \tag{5.81}$$

where we use (5.72), (5.73), (5.80) and the fact that $\det(I - e^{-i\nu}\Upsilon_\infty\Phi_\mathrm{s}(2\ell_0,0)\Upsilon_\infty) = e^{2im\nu}\mathcal{E}_{s,0}(0,e^{i\nu})$ and $\det(e^{i\nu}\Upsilon_{-\infty}\Phi_\mathrm{s}(0,2\ell_0)\Upsilon_{-\infty} - I) = \mathcal{E}_{s,0}(0,e^{i\nu})$ are bounded away from 0 by a $\nu$-independent constant for $\nu \in \mathcal{S}_\delta$.

Recall that $(f, g, h, a, b_-)$ satisfy (5.65) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon, \lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon, \lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon, \lambda)$ hold true. Moreover, by identity (5.29), $(a, b, c, d)$ satisfy (5.67), (5.68), (5.70) and (5.71) if and only if the jump $\varphi^+_{f,\varepsilon}(0, \lambda) - \varphi^-_{f,\varepsilon}(0, \lambda)$ lies in $Z^\perp$. Thus, take $c_+ := c_{\mathrm{p},+}$ and define quantities $a_\pm, b_\pm, c_-, d_\pm, f, g$ and $h$ through (5.81), where we suppress their $\varepsilon$-, $\lambda$- and $\nu$-dependence for notational convenience. Then, (5.50), (5.52) and (5.55) define for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x, \lambda)$ to (3.2) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ in the space $Z^\perp$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$.

Now, estimate (5.46) follows readily by approximating the coefficients $(a, b, c, d, f, g, h)$ in the variation of constants formulations (5.50), (5.52) and (5.55) of the solution $\varphi_{\nu,\varepsilon}(x, \lambda)$ using (5.49), (5.51), (5.53), (5.54), (5.56), (5.79) and (5.81).

Next, we show that for any $\nu \in \mathcal{S}_\delta$ the jump $J_{\nu,\varepsilon}(\lambda)$, defined in (5.48), of $\varphi_{\nu,\varepsilon}(x, \lambda)$ at $x = 0$ vanishes for a unique $\lambda$-value in $D_{\eta,\varepsilon}$. Fix $\nu \in \mathcal{S}_\delta$. The jump $J_{\nu,\varepsilon}(\lambda)$ can be expressed as the difference of the two variation of constants formulas (5.50) and (5.52) at $x = 0$ with coefficients $a_\pm, b_\pm, c_\pm$ and $d_\pm$ defined through (5.81) and $c_+ = c_{\mathrm{p},+}$. We observe that $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, because the perturbation $\mathcal{B}_{0,\varepsilon}(x, \lambda)$ and the linear maps $\mathcal{H}^{13}_{\varepsilon,\lambda}$, $\mathcal{H}^{14}_{\varepsilon,\lambda}$ and $\mathcal{H}^{15}_{\varepsilon,\lambda}$ are analytic in $\lambda$. For any $\lambda \in D_{\eta,\varepsilon}$ the jump is approximated by

$$\begin{aligned}&\left\| J_{\nu,\varepsilon}(\lambda) - d_+ + d_- - \lambda \int_\infty^0 \Phi^{uc}_{0,+}(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \lambda \int_{-\infty}^0 \Phi^{sc}_{0,-}(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy \right\| \\ &\qquad \leq C|\log(\varepsilon)|^2 \left(\varepsilon + |\lambda| \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right),\end{aligned} \tag{5.82}$$

using (5.46), (5.49), (5.72) and (5.81). By Proposition 5.1 we have $\psi_{\mathrm{ad}}(0) \in \ker(P_{f,+}(0)^*) \cap P_{f,-}(0)^*[\mathbb{C}^{2n}]$. Therefore, it holds

$$Z^\perp \subset \ker(P^s_+(0)^*) \cap \ker(P^u_-(0)^*), \tag{5.83}$$

by (5.19). The jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ of $\varphi_{\nu,\varepsilon}(x, \lambda)$ at $x = 0$ vanishes if and only if

$$\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle = 0. \tag{5.84}$$

With the aid of (5.83) we calculate

$$\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, \int_\infty^0 \Phi^{uc}_{0,+}(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \int_{-\infty}^0 \Phi^{sc}_{0,-}(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy \right\rangle = -\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x, u_0)\rangle\, dx.$$

Combining the latter with (5.82) yields for $\lambda \in D_{\eta,\varepsilon}$,

$$\left\| \left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle + \lambda \int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x, u_0)\rangle\, dx \right\| \leq C|\log(\varepsilon)|^2 \left(\varepsilon + |\lambda| \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right),$$

since $d_+ \in Z^s$ and $d_- \in Z^u$ are in the orthogonal complement of $Z^\perp$ by Proposition 5.1. Hence, because the $\lambda$- and $\varepsilon$-independent integral $\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x, u_0)\rangle\, dx$ is non-zero by Proposition 5.1 and the jump $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, Rouché's Theorem implies that equation (5.84) has, provided $\eta, \varepsilon > 0$ are sufficiently small, a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

Our last step is to prove estimate (5.47). Fix $\nu \in \mathcal{S}_\delta$. First, we establish the a priori bound

$$\left\| \varphi_{\nu,\varepsilon}(x, \lambda) - \phi'_{\mathrm{p},\varepsilon}(x) \right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in D_{\eta,\varepsilon}, \tag{5.85}$$

using (5.46) and (5.78). By subtracting (5.74) from (5.50) and (5.75) from (5.52), we obtain variation of constants formulas for $\varphi_{\nu,\varepsilon}(x, \lambda) - \phi'_{\mathrm{p},\varepsilon}(x)$ on $I^+_{f,\varepsilon}$ and $I^-_{f,\varepsilon}$, respectively. Our approach is to obtain leading-order expressions for the coefficients $a_\pm - a_{\mathrm{p},\pm}$, $b_\pm - b_{\mathrm{p},\pm}$, $c_\pm - c_{\mathrm{p},\pm}$ and $d_\pm - d_{\mathrm{p},\pm}$ in these variation of constants formulas. By (5.77), (5.79) and (5.81) it holds

$$\begin{aligned} c_\pm - c_{\mathrm{p},\pm} &= 0, \\ \left\|a_\pm - a_{\mathrm{p},\pm}\right\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad \lambda \in D_{\eta,\varepsilon}, \\ \left\|b_\pm - b_{\mathrm{p},\pm} + \mathcal{B}(\nu)\right\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2, \end{aligned} \tag{5.86}$$

where $\mathcal{B}(\nu)$ is defined in (3.13). Estimating $d_\pm - d_{\mathrm{p},\pm}$ is more elaborate. Note that the jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ lies in the kernels of $Q^u$ and $Q^s$ by (5.29). Thus, to estimate $d_+ - d_{\mathrm{p},+}$, we apply the projection $Q^s$ to

$$J_{\nu,\varepsilon}(\lambda) = \lim_{x\downarrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)\right) - \lim_{x\uparrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)\right), \quad \lambda \in D_{\eta,\varepsilon},$$

yielding

$$\begin{aligned} d_+ - d_{\mathrm{p},+} = \Phi^s_{0,-}(0,-\Xi_\varepsilon)(a_- - a_{\mathrm{p},-}) &+ \int_{-\Xi_\varepsilon}^0 \Phi^s_{0,-}(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right]dy \\ &- Q^s\int_{\Xi_\varepsilon}^0 \Phi^c_{0,+}(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right]dy \end{aligned}$$

by (5.20), (5.66) and (5.86). Therefore, (5.46), (5.49), (5.72), (5.85) and (5.86) imply

$$\left\|d_+ - d_{\mathrm{p},+}\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^2 + |\lambda|\right), \quad \lambda \in D_{\eta,\varepsilon}. \tag{5.87}$$

Subtracting (5.74) from (5.50) gives for each $\lambda \in D_{\eta,\varepsilon}$ a variation of constants formula for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)$ on $I^+_{f,\varepsilon}$:

$$\begin{aligned} \varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x) = \Phi^u_{0,+}(x,\Xi_\varepsilon)(a_+ - a_{\mathrm{p},+}) &+ \Phi_{in}(x)(b_+ - b_{\mathrm{p},+}) + \Phi^s_{0,+}(x,0)(d_+ - d_{\mathrm{p},+}) \\ &+ \int_0^x \Phi^s_{0,+}(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right]dy \\ &+ \int_{\Xi_\varepsilon}^x \Phi^{uc}_{0,+}(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right]dy, \end{aligned}$$

where we use $c_+ = c_{\mathrm{p},+}$. Applying (5.46), (5.49), (5.72), (5.85), (5.86) and (5.87) to the latter identity yields the approximation (5.47) on $[0, \Xi_\varepsilon/2]$. The proof of (5.47) on $[-\Xi_\varepsilon/2, 0]$ is analogous. □

**Remark 5.6.** The proof of Theorem 5.5 provides a Lyapunov-Schmidt type reduction procedure. By constructing the piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.2) via Lin's method, we invert a certain part of the operator $\mathcal{L}_\varepsilon - \lambda$, defined in §3, and we obtain a one-dimensional reduced equation (5.84) describing the remaining unsolved part.

Thus, solving (5.84) for $\lambda$ yields the desired simple eigenvalue $\lambda_\varepsilon(\nu)$ of $\mathcal{L}_\varepsilon$ about the origin. A leading-order expression of $\lambda_\varepsilon(\nu)$ can be obtained by calculating the leading order of the $\varepsilon$- and $\lambda$-dependent parts of (5.84). Alternatively, we use the key identity (5.2) to derive a leading-order expression for $\lambda_\varepsilon(\nu)$ – see §5.5. ■

## 5.5 Conclusion

In this subsection we finish the proof of Theorem 3.14. We concluded in §5.1 that all that remains to show is the approximation (3.11). Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and $\Xi_\varepsilon$ be as in (3.10), (5.3) and (5.4), respectively, and denote by $C > 0$ a constant, which is independent of $\varepsilon$ and $\nu$. Fix $\nu \in \mathcal{S}_\delta$. Consider the solution $\varphi_{\nu,\varepsilon}(x, \tilde{\lambda}_\varepsilon(\nu))$ to the full eigenvalue problem (3.2) on $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, established in Theorem 5.5, and define $\check{\varphi}_{\nu,\varepsilon}$ by (5.6). Clearly, $\check{\varphi}_{\nu,\varepsilon}$ is a solution to (3.2) on the whole real line. In §5.1 we showed that it holds $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ and that for $\check{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$ the key identity (5.2) is satisfied. To obtain a leading-order expression for $\lambda_\varepsilon(\nu)$ we approximate the integrals in (5.2) using Theorem 5.5.

First, Theorem 2.2 and estimate (5.46) imply that $\check{\phi}_{\nu,\varepsilon}$ and $\phi_{\mathrm{p},\varepsilon}$ are bounded on $\mathbb{R}$ by a constant independent of $\varepsilon$ and $\nu$. On the other hand, the solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint equation (3.7) satisfies

$$\|\psi_{\mathrm{ad}}(x)\| \leq Ce^{-\mu_r|x|}, \quad x \in \mathbb{R},$$

by Proposition 5.1. Thus, using estimate (5.46) we approximate

$$\left\|\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x)dx - \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x,u_0)dx\right\| \leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda_\nu(\varepsilon)|\right). \tag{5.88}$$

In addition, by estimate (5.47) and Theorem 2.2 we have

$$\begin{aligned}\left\|\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \left(\partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x,u_0),0)\right)\left(\tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x)\right)dx\right\| \\ \leq C\varepsilon|\log(\varepsilon)|^2\left(\varepsilon^2|\log(\varepsilon)|^3 + |\lambda_\varepsilon(\nu)|\right),\end{aligned} \tag{5.89}$$

where we use that $\psi_{\mathrm{ad},2}(x)$ is odd by Proposition 5.1, $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even by Theorem 2.2, $v_{\mathrm{h}}(x,u_0)$ is even by **(E1)** and the $v$-components of $\Phi_{in}(x)\mathcal{B}(\nu)$ are even by **(E1)**. Integration by parts gives

$$\begin{aligned}&\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)\left(\tilde{u}_{\nu,\varepsilon}(x) - u'_{\mathrm{p},\varepsilon}(x)\right)dx \\ &\qquad = -\varepsilon\int_{-\infty}^{\infty}\int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(y),\varepsilon) D_1^{-1}\left(\tilde{p}_{\nu,\varepsilon}(x) - p'_{\mathrm{p},\varepsilon}(x)\right)dydx,\end{aligned}$$

since $\psi_{\mathrm{ad},2}(x)$ is odd and $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even. Applying estimate (5.47) and Theorem 2.2 to the latter yields

$$\begin{aligned}&\left\|\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)\left(\tilde{u}_{\nu,\varepsilon}(x) - u'_{\mathrm{p},\varepsilon}(x)\right)dx \right.\\ &\qquad\left. -\varepsilon^2\int_{-\infty}^{\infty}\int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(u_0, v_{\mathrm{h}}(y,u_0),0)dydxB(\nu)\right\| \\ &\quad \leq C\varepsilon|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^3 + |\lambda_\nu(\varepsilon)|\right),\end{aligned} \tag{5.90}$$

with $B(\nu)$ defined in (3.13), where we use $\psi_{\mathrm{ad},2}(x)$ is odd, $v_{\mathrm{h}}(x,u_0)$ is even and the $p$-component of $(I - \Phi_{in}(x))\mathcal{B}(\nu)$ is odd by **(E1)**. Finally, since the integral $\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x,u_0)dx$ is non-zero by Proposition 5.1, the key identity (5.2) in combination with the estimates (5.88), (5.89) and (5.90) gives

$$\left\|\lambda_\varepsilon(\nu) + \varepsilon^2 \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(u_0, v_{\mathrm{h}}(y,u_0),0)dydxB(\nu)}{\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x,u_0)dx}\right\| \leq C\varepsilon^3|\log(\varepsilon)|^5.$$

The latter yields the approximation (3.11) of $\lambda_\varepsilon(\nu)$ by switching the order of integration in the numerator using that $\psi_{\mathrm{ad},2}$ is odd and $v_{\mathrm{h}}(x,u_0)$ is even. This concludes the proof of Theorem 3.14. □

**Remark 5.7.** In the proof of Theorem 3.14 we have obtained for any $\nu \in \mathcal{S}_\delta$ an eigenfunction

$$\psi_{\nu,\varepsilon}(\check{x}) := \begin{pmatrix} \tilde{u}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x}) \\ \tilde{v}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x}) \end{pmatrix} e^{i\nu\check{x}/2\ell_\varepsilon} \in H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n}),$$

corresponding to the eigenvalue $\lambda_\varepsilon(\nu)$ of the operator $\mathcal{L}_{\nu,\varepsilon}$ defined in §3. The approximations in Theorem 5.5 and its proof provide leading-order control over this eigenfunction. We observe that $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by $(0, \partial_x v_{\mathrm{h}}(\varepsilon^{-1}\check{x}, u_0))$ along the pulse, i.e. for $\varepsilon\check{x} \in I_{f,\varepsilon}$. The derivative $\partial_x v_{\mathrm{h}}(x,u_0)$ corresponds to the translational eigenfunction at $\lambda = 0$ of the linearization of $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_h(x, u_0)$. Thus, along the pulse, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is independent of $\nu$. On the other hand, along the slow manifold, i.e. for $\varepsilon\check{x} \in I_{s,\varepsilon}$, $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by the $u$-components of

$$2\varepsilon e^{i\nu\check{x}/2\ell_\varepsilon}\Phi_s(\check{x},0)\Upsilon_0\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0,0)\Upsilon_0\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix},$$

by (5.40), (5.54), (5.55), (5.56), (5.64), (5.65) and (5.81), where $\mathcal{J}$ is given by (2.7), $\Phi_s(\check{x},\check{y})$ is the evolution (2.9) and $\Upsilon_0$ is defined in (3.13). Thus, along the slow manifold, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is dictated by the slow variational equation (2.9) and the value of $\nu$. ■

# A Exponential di- and trichotomies

Exponential dichotomies enable us to track solutions in linear systems by separating the solution space in solutions that either decay exponentially in forward time or else in backward time. Moreover, the associated projections inherit analytic dependence of the system on a spectral parameter $\lambda$. Therefore, they provide a natural framework [42] to capture the linear dynamics of eigenvalue problems arising in our spectral stability analysis. For an extensive introduction on dichotomies the reader is referred to [6, 40].

**Definition A.1.** Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x,y)$ the evolution operator of

$$\varphi_x = A(x)\varphi, \quad \varphi \in \mathbb{C}^n. \tag{A.1}$$

Equation (A.1) has *an exponential dichotomy on $J$ with constants $K, \mu > 0$ and projections $P(x)\colon \mathbb{C}^n \to \mathbb{C}^n$* if for all $x, y \in J$ it holds

- $P(x)T(x,y) = T(x,y)P(y)$;
- $\|T(x,y)P(y)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x,y)(I - P(y))\| \leq Ke^{-\mu(y-x)}$ for $y \geq x$.

Let $P(x)$ be the family of projections associated with an exponential dichotomy on $J$. For each $x, y \in J$, we denote by $T^s(x,y) = T(x,y)P(y)$ and $T^u(x,y) = T(x,y)(I - P(y))$ the *stable and unstable evolution* of system (A.1), leaving the projection $P(y)$ implicit.

A generalization of the concept of exponential dichotomies is the notion of *exponential trichotomies* to capture linear systems that exhibit center behavior in addition to exponential decay in forward and backward time. The slow-fast structure in the eigenvalue problem (3.2) naturally induces such behavior. We employ the following definition.

**Definition A.2.** Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x,y)$ the evolution operator of (A.1). Equation (A.1) has an *exponential trichotomy* on $J$ with constants $K, \mu > 0$ and projections $P^u(x), P^s(x), P^c(x)\colon \mathbb{C}^n \to \mathbb{C}^n$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) + P^c(x) = I$;
- $P^{u,s,c}(x)T(x,y) = T(x,y)P^{u,s,c}(y)$;
- $\|T(x,y)P^s(y)\|, \|T(y,x)P^u(x)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x,y)P^c(y)\| \leq K$.

We often use the abbreviations $T^{u,s,c}(x,y) = T(x,y)P^{u,s,c}(y)$ leaving the associated projections of the exponential trichotomy implicit.

## A.1 Exponential dichotomies for slowly varying systems

Clearly, an autonomous linear system $\varphi_x = A_0\varphi$, where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is hyperbolic, admits an exponential dichotomy on $\mathbb{R}$. The associated dichotomy projection is given by the spectral projection onto the stable eigenspace of $A_0$. If $A_0$ depends analytically on a parameter $\lambda$, then the spectral projection inherits the analyticity. This can be extended to non-autonomous systems of the form

$$\varphi_x = A(x,\lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{A.2}$$

depending analytically on a spectral parameter $\lambda$. If $A(x,\lambda)$ varies slowly and is pointwise hyperbolic, then (A.1) admits an exponential dichotomy that has analytic projections close to the spectral projections onto the stable eigenspace of $A(x,\lambda)$.

The latter result is proved in [5, Proposition 6.5] in the setting of the FitzHugh-Nagumo system. In addition, in [6, Proposition 6.1] the result is proved for general systems of the form (A.1). However, the result in [6] lacks the desired closeness estimates on the dichotomy projections and analytic dependence on parameters is not shown. Therefore, we provide a proof of these two facts along the lines of [5, Proposition 6.5].

**Proposition A.3.** *Let $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ with $b - a > 2$ and $\Omega \subset \mathbb{C}$ open. Denote $X = [a,b] \times \Omega$ and let $A \in C^1(X, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Assume that $A(x,\cdot)$ is analytic on $\Omega$ for each $x \in [a,b]$ and that there exists constants $\alpha > 0$ and $M > 1$ such that:*

*i. For each $(x,\lambda) \in X$ the matrix $A(x,\lambda)$ is hyperbolic with spectral gap larger than $\alpha$;*

*ii. The matrix function $A$ is bounded by $M$ on $X$.*

*There exists $\delta > 0$, depending only on $\alpha$ and $M$, such that, if we have*

$$\sup_{(x,\lambda)\in X} \|\partial_x A(x,\lambda)\| \leq \delta,$$

*then (A.2) has an exponential dichotomy on $[a+1, b-1]$ for any $\lambda \in \Omega$ with constants $C, \mu > 0$ and projections $P(x,\lambda)$ such that $P(x,\cdot)$ is analytic on $\Omega$ for each $x \in [a+1, b-1]$. In addition, we have $\mu = \frac{1}{2}\alpha$ and $C$ depends only on $M$, $\alpha$ and $n$. Finally, for any $(y,\lambda) \in [a+1, b-1] \times \Omega$ we have the estimate*

$$\|P(y,\lambda) - \mathcal{P}(y,\lambda)\| \leq C \sup_{(x,\lambda)\in X} \|\partial_x A(x,\lambda)\|, \tag{A.3}$$

*where $\mathcal{P}(x,\lambda)$ is the spectral projection onto the stable eigenspace of $A(x,\lambda)$.*

**Proof.** In the following, we denote by $C > 0$ a constant depending only on $M$, $n$ and $\alpha$.

Our approach is to extend system (A.2) to the whole real line, such that it varies only on the finite interval $[a,b]$. We establish an exponential dichotomy for this extended system using [6, Proposition 6.1]. The range or kernel of the dichotomy projections must be analytic for $x \in \mathbb{R} \setminus [a,b]$ by analyticity of the spectral projections. These analyticity properties can be interpolated to the interval $[a,b]$. Finally, to prove the closeness estimate (A.3), we approximate the stable evolution operator of system (A.2) by $\mathcal{P}(x,\lambda)\exp(A(x,\lambda)(x-y))$, using that the derivative of $A(x,\lambda)$ is small.

We introduce a smooth partition of unity $\chi_i \colon \mathbb{R} \to [0,1]$, $i = 1,2,3$ satisfying

$$\sum_{i=1}^{3} \chi_i(x) = 1, \quad |\chi_2'(x)| \leq 2, \quad x \in \mathbb{R},$$

$$\mathrm{supp}(\chi_1) \subset (-\infty, a+1), \quad \mathrm{supp}(\chi_2) \subset (a,b), \quad \mathrm{supp}(\chi_3) \subset (b-1, \infty).$$

The equation

$$\varphi_x = \mathcal{A}(x,\lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{A.4}$$

with

$$\mathcal{A}(x,\lambda) := \chi_1(x)A(a,\lambda) + \chi_2(x)A(x,\lambda) + \chi_3(x)A(b,\lambda),$$

coincides with (A.2) on $[a+1, b-1]$. We calculate

$$\partial_x\mathcal{A}(x,\lambda) = \begin{cases} \chi_2(x)\partial_x A(x,\lambda), & x \in (a+1, b-1), \\ \chi_2'(x)(A(x,\lambda) - A(a,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [a, a+1], \\ \chi_2'(x)(A(x,\lambda) - A(b,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [b-1, b], \\ 0, & \text{otherwise}. \end{cases}$$

First, we have $\|\partial_x\mathcal{A}(x,\lambda)\| \leq 3\delta$ for each $(x,\lambda) \in \mathbb{R}\times\Omega$ by the mean value theorem. Second, by the spectral estimates in [34] the Hausdorff distance between the spectra of $A(a,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for each $(x,\lambda) \in (-\infty, a+1]\times\Omega$. Similarly, the Hausdorff distance between the spectra of $A(b,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for every $(x,\lambda) \in [b-1,\infty)\times\Omega$. Hence, for $\delta > 0$ sufficiently small, the matrix $\mathcal{A}(x,\lambda)$ is hyperbolic for each $(x,\lambda) \in \mathbb{R}\times\Omega$ with spectral gap larger than $\frac{1}{2}\alpha$. Third, $\mathcal{A}$ is bounded by $M$ on $\mathbb{R}\times\Omega$. Combining these three items with [6, Proposition 6.1] implies that system (A.4) admits, provided $\delta > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$ – with $\mu = \frac{1}{2}\alpha$ – and projections $P(x,\lambda)$.

The next step is to prove that the projections $P(x,\cdot)$ are analytic in $\Omega$ for each $x \in \mathbb{R}$. Any solution to the constant coefficient system $\psi_x = A(a,\lambda)\psi$ that converges to 0 as $x \to -\infty$ must be in the kernel of the spectral projection $\mathcal{P}(a,\lambda)$ onto the stable eigenspace of $A(a,\lambda)$. Hence, it holds $\ker(\mathcal{P}(a,\lambda)) = \ker(P(a,\lambda))$ by construction of (A.4). Moreover, the spectral projection $\mathcal{P}(a,\cdot)$ is analytic on $\Omega$, since $A(a,\cdot)$ is analytic on $\Omega$. Thus, $\ker(P(a,\lambda))$ and similarly $P(b,\lambda)[\mathbb{C}^n]$ must be analytic subspaces – see [22, Chapter 18] – in $\lambda \in \Omega$. Denote by $T(x,y,\lambda)$ the evolution operator of (A.4), which is by [28, Lemma 2.1.4] analytic in $\lambda \in \Omega$ for each $x, y \in \mathbb{R}$. We conclude that both $\ker(P(a,\lambda))$ and $P(a,\lambda)[\mathbb{C}^n] = T(a,b,\lambda)P(b,\lambda)[\mathbb{C}^n]$ are analytic subspaces in $\lambda \in \Omega$. Therefore, the projection $P(a,\cdot)$ – and thus any projection $P(x,\cdot) = T(x,a,\cdot)P(a,\cdot)$, $x \in \mathbb{R}$ – is analytic in $\Omega$.

Finally, we prove that the projections $P(x,\lambda)$ can be approximated by the spectral projections $\mathcal{P}(x,\lambda)$ onto the stable eigenspace of $\mathcal{A}(x,\lambda)$ for any $(x,\lambda) \in \mathbb{R}\times\Omega$. Define $\delta_* := \sup\{\|\partial_x A(x,\lambda)\| : (x,\lambda) \in X\} > 0$. Take $z \in \mathbb{R}$ and $v \in \mathcal{P}(z,\lambda)[\mathbb{C}^n]$. Observe that

$$\hat{\varphi}(x,\lambda) := \mathcal{P}(x,\lambda)e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}v, \quad (x,\lambda) \in \mathbb{R}\times\Omega,$$

satisfies the inhomogeneous equation

$$\varphi_x = \mathcal{A}(x,\lambda)\varphi + g(x,\lambda),$$

with

$$g(x,\lambda) := e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}\left[\partial_x\left(\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)\right)(x-z) + \partial_x\mathcal{P}(x,\lambda)\right]v.$$

By uniformity of the bound on the spectral gap of $\mathcal{A}$, there exists a contour $\Gamma \subset \mathbb{C}$, depending only on $M, \alpha$ and $n$, containing precisely those eigenvalues of $\mathcal{A}(x,\lambda)$ of negative real part for all $(x,\lambda) \in \mathbb{R}\times\Omega$. Thus, we have

$$\mathcal{P}(x,\lambda) = \frac{1}{2\pi i}\oint_\Gamma (w - \mathcal{A}(x,\lambda))^{-1}dw, \quad (x,\lambda) \in \mathbb{R}\times\Omega. \tag{A.5}$$

By [21, Corollary 1.2.4] the norm of the resolvent $(w - \mathcal{A}(x,\lambda))^{-1}$ can be bounded in terms of $M, n$ and the distance $d(w, \sigma(\mathcal{A}(x,\lambda)))$. Hence, choosing the contour $\Gamma$ appropriately, we observe

$$\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\mathcal{P}(x,\lambda)\| \leq C. \tag{A.6}$$

Since $\mathcal{P}(x,\lambda)$ is the projection onto the stable eigenspace of $\mathcal{A}(x,\lambda)$ and $\mathcal{A}$ is uniformly bounded by $M$ on $\mathbb{R}\times\Omega$ and has a uniform spectral gap larger than $\mu = \frac{1}{2}\alpha$ on $\mathbb{R}\times\Omega$, we have by [21, Theorem 1.2.1] the bound

$$\sup_{\lambda\in\Omega} \left\| e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)} \right\| \leq C e^{-\mu(x-z)}, \quad x \geq z. \tag{A.7}$$

Differentiating identity (A.5) yields

$$\partial_x \mathcal{P}(x,\lambda) = \frac{1}{2\pi i} \oint_\Gamma (w - \mathcal{A}(x,\lambda))^{-1} \partial_x \mathcal{A}(x,\lambda)(w - \mathcal{A}(x,\lambda))^{-1} dw,$$

for each $(x,\lambda) \in \mathbb{R}\times\Omega$. Since the norm of the resolvent $(w - \mathcal{A}(x,\lambda))^{-1}$ can be bounded in terms of $M, n$ and $d(w, \sigma(\mathcal{A}(x,\lambda)))$, we observe that $\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\partial_x \mathcal{P}(x,\lambda)\| \leq C\delta_*$. Thus, combining the latter with (A.6) and (A.7) yields

$$\sup_{\lambda\in\Omega} \|g(x,\lambda)\| \leq C\delta_* \|v\|, \quad x \geq z. \tag{A.8}$$

Take $\xi = z - \log(\delta_*)\mu^{-1} \geq z$. By the variation of constants formula there exists $w \in \mathbb{C}^3$ such that

$$\hat{\varphi}(x,\lambda) = T(x,\xi,\lambda)w + \int_z^x T^s(x,y,\lambda)g(y,\lambda)dy + \int_\infty^x T^u(x,y,\lambda)g(y,\lambda)dy, \tag{A.9}$$

for $x \geq z$ and $\lambda \in \Omega$. Evaluating (A.9) at $x = \xi$, while using (A.6), (A.7) and (A.8), we derive $\|w\| \leq C\delta_*\|v\|$. Thus, applying $I - P(z,\lambda)$ to (A.9) at $x = z$, yields the bound $\|(I - P(z,\lambda))v\| \leq C\delta_*\|v\|$ for every $v \in \mathcal{P}(z,\lambda)[\mathbb{C}^n]$ by (A.7) and (A.8). Similarly, one shows that for every $v \in \ker(\mathcal{P}(z,\lambda))$ we have $\|P(z,\lambda)v\| \leq C\delta_*\|v\|$. Thus, we obtain for any $(z,\lambda) \in \mathbb{R}\times\Omega$

$$\|[P - \mathcal{P}](z,\lambda)\| \leq \|[(I - P)\mathcal{P}](z,\lambda)\| + \|[P(I - \mathcal{P})](z,\lambda)\| \leq C\delta_*.$$

Since (A.4) coincides with (A.2) on $[a+1, b-1]$, we have established (A.3), which concludes the proof. □

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