Spectral stability of periodic wave trains in singularly perturbed RD systems

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Class of reaction-diffusion systems

A **singularly perturbed** (ε small), **multicomponent, reaction-diffusion system** takes the form

$$\varphi_t = \underbrace{D(\varepsilon)}_{\text{diagonal}}\varphi_{yy} + f(\varphi,\varepsilon), \quad \varphi \in \mathbb{R}^k.$$

Our model is given by

$$\begin{cases} u_t &= D_1 u_{yy} - H_1(u,\varepsilon) - \varepsilon^{-1} H_2(u,v,\varepsilon)v, \\ v_t &= \varepsilon^2 D_2 v_{yy} - G(u,v,\varepsilon) \end{cases}, \quad (1)$$

$$u \in \mathbb{R}^m, v \in \mathbb{R}^n, D_{1,2} \text{ diagonal}.$$

Rescaling $y = \varepsilon x$ gives

$$\begin{cases} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u,\varepsilon) - \varepsilon H_2(u,v,\varepsilon)v \\ v_t &= D_2 v_{xx} - G(u,v,\varepsilon) \end{cases}. \quad (2)$$

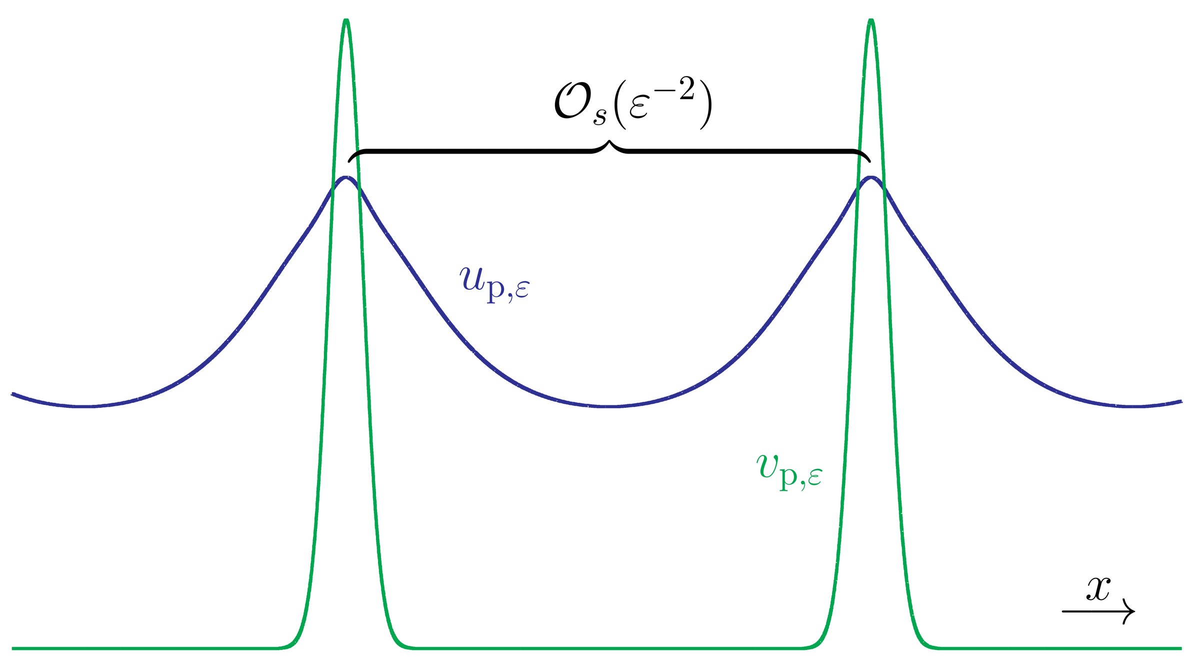
Motivation for studying this model:

- It includes the **Gierer-Meinhardt equations**;
- Although our stability analysis works for **any nonlinearity** $H_1(u,\varepsilon) - \varepsilon^{-\eta} H_2(u,v,\varepsilon)v, \eta \in [0,1]$, in the u-component of (1), the choice $\eta \neq 1$ renders only **unstable** periodic pulses.

Solutions under consideration

We consider **stationary, spatially periodic, wave train solutions** $\gamma_{p,\varepsilon}(x)$ to (2) that consist of:

- **Fast** excursions close to $(u_0, v_h(x))$, where $v_h(x)$ is a **pulse** solution to $D_2 v_{xx} = G(u_0, v, 0)$;
- **Slow** excursions close to $(u_s(y), 0)$, where $u_s(y)$ is a solution to $D_1 u_{yy} = H_1(u, 0)$.



To **construct** $\gamma_{p,\varepsilon}$ use geometric singular perturbation theory with appropriate **exchange lemmas**.

Analysis of the spectrum

Step 1: formulate as eigenvalue problem

- **Linearizing** (1) around $\gamma_{p,\varepsilon}$ in $C_{ub}(\mathbb{R}, \mathbb{R}^{n+m})$ results in a parabolic, periodic differential operator

$$\mathcal{L}_\varepsilon[\varphi] = D(\varepsilon)\varphi_{yy} + \partial f(\gamma_{p,\varepsilon}(y),\varepsilon)\varphi.$$

- A point λ is in the **spectrum** $\sigma(\mathcal{L}_\varepsilon)$ iff there exists a solution $\varphi \in C^2_{ub}(\mathbb{R}, \mathbb{C}^{n+m}) \setminus \{0\}$ to the **eigenvalue problem** $\mathcal{L}_\varepsilon[\varphi] = \lambda\varphi$.
- Write the problem $\mathcal{L}_\varepsilon[\varphi] = \lambda\varphi$ as a **first order ODE**

$$\psi_x = \begin{pmatrix} \varepsilon\mathcal{A}_{11,\varepsilon}(x,\lambda) & \varepsilon\mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x,\lambda) \end{pmatrix}\psi. \quad \text{(EVP)}$$

⇓

Step 2: define Evans function

- An **Evans function** is an **analytic** map that locates the spectrum.
- By **Floquet theory**: (EVP) has a bounded solution iff there exists $\gamma \in S^1$ such that

$$\underbrace{D_\varepsilon(\lambda,\gamma)}_{\text{Evans function}} = \det(\underbrace{\mathcal{T}_\varepsilon(0,-L_\varepsilon,\lambda)}_{\text{Evolution of (EVP)}} - \gamma\mathcal{T}_\varepsilon(0,L_\varepsilon,\lambda)) = 0.$$

⇓

Step 3: take singular limit of (EVP)

- The **analytic** operator $\mathcal{F}(\lambda)$ on $C_{ub}(\mathbb{R}, \mathbb{C}^{2n})$ given by

$$\mathcal{F}(\lambda)[\omega] = \omega_x - \mathcal{A}_{22,0}(x,\lambda)\omega,$$

corresponds to the (fast) singular limit of (EVP).

- There exists $\Lambda > 0$ such that $\mathcal{F}(\lambda)$ is **Fredholm of index** 0 for

$$\lambda \in \mathcal{C}_\Lambda := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\Lambda\}.$$

- Let $D_f : \mathcal{C}_\Lambda \to \mathbb{C}$ be an (analytic) Evans function, which detects the (finite set of) **eigenvalues** of the operator pencil $\mathcal{F}$ on $\mathcal{C}_\Lambda$.

⇓

Step 4: factorize via Riccati transform

- Define for $\delta > 0$ the set

$$\Sigma_{\Lambda,\delta} := \mathcal{C}_\Lambda \setminus \underbrace{\mathcal{N}(D_f^{-1}(0),\delta)}_{\delta\text{-neighborhood}}.$$

Theorem (Chang [2]). *Let $\delta > 0$. There exists, for $\varepsilon > 0$ sufficiently small and $\lambda \in \Sigma_{\Lambda,\delta}$, a coordinate transform $\psi = H_\varepsilon(x,\lambda)\left[\begin{smallmatrix}\chi \\ \omega\end{smallmatrix}\right]$, which* ***diagonalizes*** *system (EVP) to*

$$\chi_x = \varepsilon(\mathcal{A}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda))\chi, \quad (3)$$

$$\omega_x = (\mathcal{A}_{22,\varepsilon}(x,\lambda) - \varepsilon U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x))\omega, \quad (4)$$

where $U_\varepsilon(x,\lambda)$ is close to $\mathcal{F}(\lambda)^{-1}[\mathcal{A}_{21,0}](x)$ in ε.

Corollary. *The Evans function* ***factorizes*** *on $\Sigma_{\Lambda,\delta}$ as*

$$D_\varepsilon(\lambda,\gamma) = D_{s,\varepsilon}(\lambda,\gamma)D_{f,\varepsilon}(\lambda,\gamma),$$

where

$$D_{\substack{s\\f},\varepsilon}(\lambda,\gamma) := \det(\underbrace{\mathcal{T}_{\substack{s\\f},\varepsilon}(0,-L_\varepsilon,\lambda)}_{\text{evolution of (3)/(4)}} - \gamma\mathcal{T}_{\substack{s\\f},\varepsilon}(0,L_\varepsilon,\lambda)).$$

⇓

Step 5: approximate $D_{f,\varepsilon}$ and $D_{s,\varepsilon}$

- Using [5] we approximate on $\Sigma_{\Lambda,\delta} \times S^1$

$$D_{f,\varepsilon}(\lambda,\gamma)\underbrace{h_\varepsilon(\lambda)}_{\text{exponentially small}} = (-\gamma)^n D_f(\lambda) + \mathcal{O}(\varepsilon^{\mu_f}). \quad (5)$$

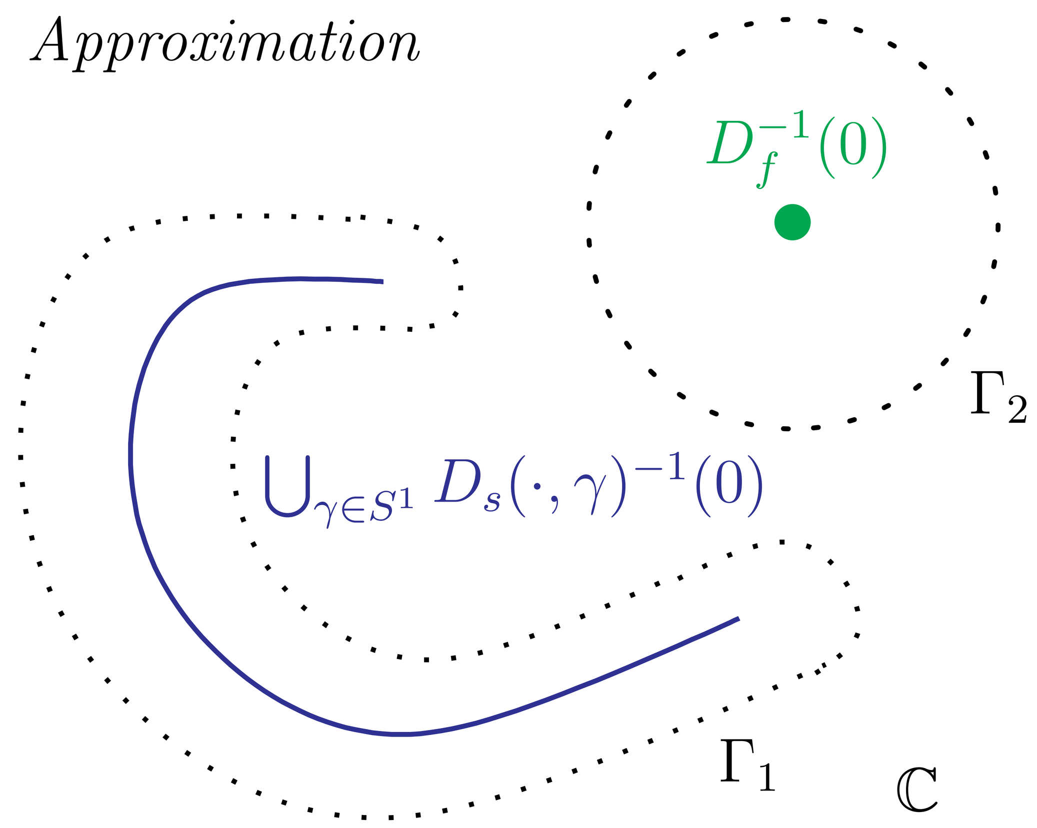
- We approximate on $\Sigma_{\Lambda,\delta} \times S^1$

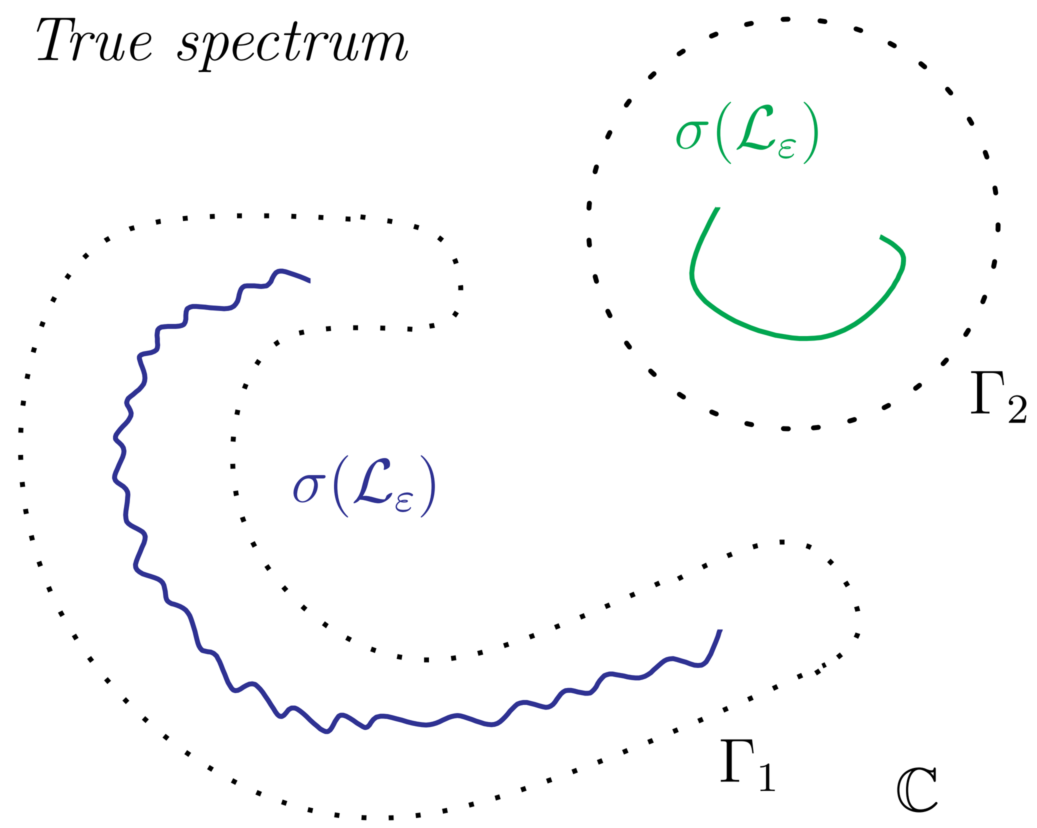
$$D_{s,\varepsilon}(\lambda,\gamma) = D_s(\lambda,\gamma) + \mathcal{O}(\varepsilon^{\mu_s}), \quad (6)$$

where $D_s(\cdot,\gamma)$ is analytic on $\mathcal{C}_\Lambda \setminus D_f^{-1}(0)$ and **meromorphic** on $\mathcal{C}_\Lambda$. Furthermore, $D_s(\lambda,\gamma)$ is an **explicit expression** in terms of $\mathcal{F}(\lambda)^{-1}[\mathcal{A}_{21,0}](x)$ and solutions to the equation $D_1 u_{yy} = (\partial_u H_1(u_s(y),0) + \lambda)u$.

Outcome of our spectral analysis

By applying **winding number arguments** to estimates (5)-(6) the **critical spectrum** $\sigma(\mathcal{L}_\varepsilon) \cap \mathcal{C}_\Lambda$ is **approximated rigorously** (with ε-continuous error).





Relevance of our method

- **Generalizes** and **formalizes** existing results (e.g. [4, 6]) to a **multi-dimensional setting** and a **broader class** of systems;
- Offers an **analytic alternative** for the **geometric** ('elephant trunk') method developed in [1, 3].

Further research

- By **translational invariance** we have $0 \in D_f^{-1}(0)$. A detailed analysis of **'small' spectrum** around 0 is needed to decide upon **spectral stability**;
- **Nonlinear stability**;
- Connection to **homoclinic limit** (Hopf dance);
- Generalization to models with **convective** terms.

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