Evans function factorization via the Riccati transform*

CT2: Evans functions

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Outline

- Background: what is the Evans function?
- The Evans function in singularly perturbed problems
  $\rightarrow$ small parameter induces factorization Evans function
- Geometric factorization results in current literature
- Analytic alternative that generalizes and formalizes factorization procedure
- Employ new factorization method
  $\rightarrow$ to standing wave trains in reaction-diffusion (RD) equations

Background: set-up

Travelling wave solution $u_*(x,t) = \varphi(x - ct)$ to

$$u_t = P(\partial_x)u + N(u), \quad u \in \mathbb{R}^n.$$

In co-moving frame $\zeta = x - ct$, linearization about φ is

$$\mathcal{L}u = P(\partial_\zeta)u + c\partial_\zeta u + \partial_u N(\varphi(\zeta))u.$$

Spectrum $\sigma(\mathcal{L})$ is determined by problem $\mathcal{L}u = \lambda u$.

Write $\mathcal{L}u = \lambda u$ as ODE

$$u_\zeta = \mathcal{A}(\zeta, \lambda)u. \qquad \text{(LSP)}$$

Background: Evans function

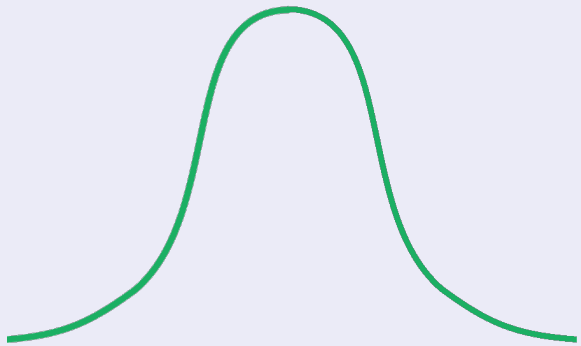
$$u_\zeta = \mathcal{A}(\zeta, \lambda)u. \qquad \text{(LSP)}$$

Definition

Evans function: analytic map that vanishes on the spectrum.

If φ is a pulse

Asymptotic matrix $\mathcal{A}^\infty(\lambda) = \lim_{\zeta \to \pm\infty} \mathcal{A}(\zeta, \lambda)$.



Essential spectrum: $\lambda \in \mathbb{C}$ for which $\mathcal{A}^\infty(\lambda)$ is non-hyperbolic.

Point spectrum: $\lambda \in \mathbb{C}$ for which

$$D(\lambda) = \det(S(\lambda)|U(\lambda)) = 0.$$

$S(\lambda), U(\lambda)$ analytic bases of initial conditions yielding bounded solutions for $\zeta \to \infty$ or $\zeta \to -\infty$, respectively.

Background: Evans function

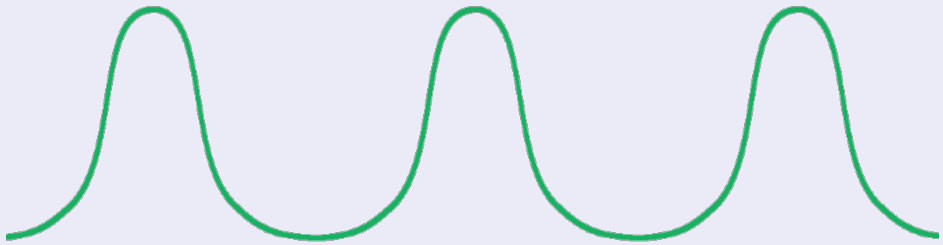
$$u_\zeta = \mathcal{A}(\zeta, \lambda)u. \tag{LSP}$$

Definition

Evans function: analytic map that vanishes on the spectrum.

If φ is a wave train

$\mathcal{A}(\cdot, \lambda)$ is $2L$-periodic.



Essential spectrum: $\lambda \in \mathbb{C}$ for which there exists $\gamma \in S^1$ such that

$$D(\lambda, \gamma) = \det(\mathcal{T}(0, L, \lambda) - \gamma \mathcal{T}(0, -L, \lambda)) = 0.$$

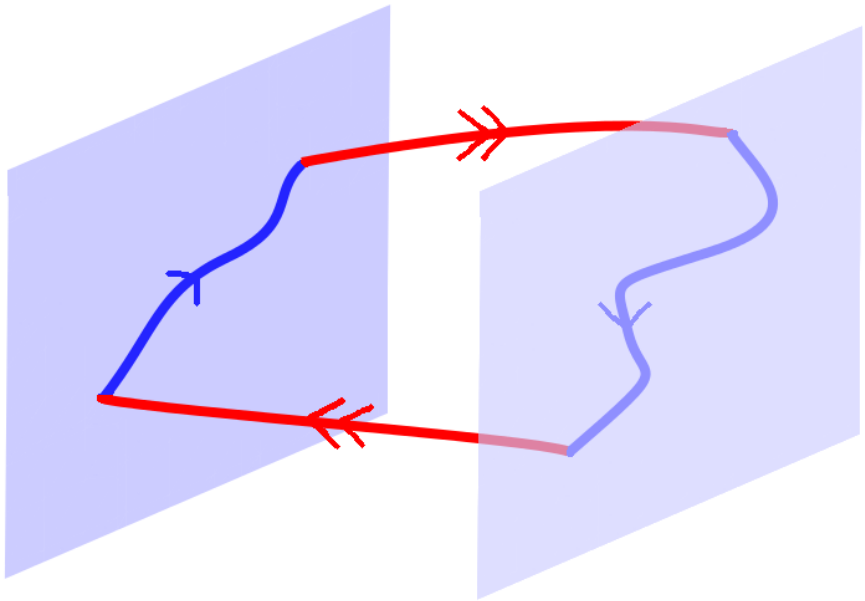
$\mathcal{T}(\zeta, \tilde{\zeta}, \lambda)$ evolution of (LSP).

Evans function in singularly perturbed problems

Small parameter $0 < \varepsilon \ll 1$ in PDE

$$u_t = P_\varepsilon(\partial_x)u + N_\varepsilon(u).$$

Construction of travelling wave close to concatenation of 'slow' and 'fast' segments (**Fenichel theory**).



Then, (LSP) often takes the form

$$u_\zeta = \mathcal{A}_\varepsilon(\zeta,\lambda)u, \quad \mathcal{A}_\varepsilon(\zeta,\lambda) = \begin{pmatrix} \varepsilon\mathcal{A}_{11,\varepsilon} & \varepsilon\mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix}[\zeta,\lambda], \qquad \text{(LSP)}$$

with $\mathcal{A}_\varepsilon(\cdot,\lambda)$ bounded ε-uniformly.

Evans function in singularly perturbed problems

$$u_\zeta = \begin{pmatrix} \varepsilon \mathcal{A}_{11,\varepsilon} & \varepsilon \mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix} [\zeta, \lambda]\ u. \qquad \text{(LSP)}$$

Two key observations in current literature:

Observation 1: factorization

Factorization Evans function in **fast** and **slow** components

$$D_\varepsilon(\lambda) = D_{f,\varepsilon}(\lambda) D_{s,\varepsilon}(\lambda), \quad 0 < \varepsilon \ll 1.$$

$D_{f/s,\varepsilon}(\lambda)$ correspond to **lower-dimensional** problems.

Rigorous **geometric** proofs for

- 1-pulses in FHN and predator-prey system [Alexander, Gardner, Jones '90]
- Wave trains in FHN system [Eszter, Gardner '99]
- Fronts in Fabry-Pérot model [Rubin '99]
- 1-pulses in Gray-Scott [Doelman, Gardner, Kaper '02]

Evans function in singularly perturbed problems

$$u_\zeta = \begin{pmatrix} \varepsilon \mathcal{A}_{11,\varepsilon} & \varepsilon \mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix} [\zeta, \lambda]\ u. \qquad \text{(LSP)}$$

Two key observations:

Observation 2: passage to singular limit

Factorization $D_\varepsilon(\lambda) = D_{f,\varepsilon}(\lambda) D_{s,\varepsilon}(\lambda)$ reveals singular limit structure

$$D_0(\lambda) = D_{f,0}(\lambda) D_{s,0}(\lambda),$$

with $D_{f,0}$ **analytic** and $D_{s,0}$ **meromorphic**.

Rigorous **geometric** proofs for

- 1-pulses in Gray-Scott model [Doelman, Gardner, Kaper '02]
- Wave trains in Gierer-Meinhardt equations [Doelman, vd Ploeg '05]
- Multi-fronts in 3-component RD system [Doelman, v Heijster, Kaper '08]
- 1-pulses in general 2-dimensional RD system [Doelman, Veerman '13]

Goal of talk

1. Present **analytic** alternative for geometric factorization procedure that works in **general, multi-dimensional** setting.

   $\rightarrow$ focus on pulses and wave trains.

2. Employ method for **standing wave trains** in a general, multi-dimensional class of RD systems.

Goal 1: analytic alternative for factorization procedure

Key of factorization

$$u_\zeta = \begin{pmatrix} \varepsilon\mathcal{A}_{11,\varepsilon} & \varepsilon\mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix}[\zeta,\lambda]\ u. \tag{LSP}$$

Suppose there exists $\Sigma \subset \mathbb{C}$ such that for all $\lambda \in \Sigma$

$$v_\zeta = \mathcal{A}_{22,\varepsilon}(\zeta,\lambda)v,$$

has an **exponential dichotomy** on $\mathbb{R}$ with constants independent of ε, λ.

Riccati transform [Chang '69], [dRDR '14]

For $0 < \varepsilon \ll 1$ and $\lambda \in \Sigma$ the **Riccati transform** $\omega = H_\varepsilon(\zeta,\lambda)u$ **diagonalizes** (LSP) to

$$\omega_\zeta = \begin{pmatrix} \varepsilon(\mathcal{A}_{11,\varepsilon} + \mathcal{A}_{12,\varepsilon}U_\varepsilon) & 0 \\ 0 & \mathcal{A}_{22,\varepsilon} - \varepsilon U_\varepsilon \mathcal{A}_{12,\varepsilon} \end{pmatrix}[\zeta,\lambda]\ \omega, \tag{DLSP}$$

where $U_\varepsilon(\zeta,\lambda)$ satisfies

$$U_\zeta = \mathcal{A}_{22,\varepsilon}(\zeta,\lambda)U + \mathcal{A}_{21,\varepsilon}(\zeta,\lambda) + \mathcal{O}(\varepsilon).$$

Factorization for $2L$-periodic wave trains

$$u_\zeta = \begin{pmatrix} \varepsilon\mathcal{A}_{11,\varepsilon} & \varepsilon\mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix}[\zeta,\lambda]\ u. \qquad \text{(LSP)}$$

1. Transformation $H_\varepsilon(\zeta,\lambda)$ is unimodalar and $2L$-periodic.
2. Evolution of (DLSP) is $\mathcal{T}_{d,\varepsilon}(\zeta,\tilde{\zeta},\lambda) = \mathrm{diag}(\mathcal{T}_{s,\varepsilon}(\zeta,\tilde{\zeta},\lambda),\mathcal{T}_{f,\varepsilon}(\zeta,\tilde{\zeta},\lambda))$.

Factorization:

$$\begin{aligned} D_\varepsilon(\lambda,\gamma) &= \det(\mathcal{T}_\varepsilon(0,L,\lambda) - \gamma\mathcal{T}_\varepsilon(0,-L,\lambda)) \\ &= \det(H_\varepsilon(0,\lambda)^{-1})\det(\mathcal{T}_{d,\varepsilon}(0,L,\lambda) - \gamma\mathcal{T}_{d,\varepsilon}(0,-L,\lambda))\det(H_\varepsilon(L,\lambda)) \\ &= \det(\mathcal{T}_{s,\varepsilon}(0,L,\lambda) - \gamma\mathcal{T}_{s,\varepsilon}(0,-L,\lambda))\det(\mathcal{T}_{f,\varepsilon}(0,L,\lambda) - \gamma\mathcal{T}_{f,\varepsilon}(0,-L,\lambda)) \\ &= D_{s,\varepsilon}(\lambda,\gamma)D_{f,\varepsilon}(\lambda,\gamma) \end{aligned}$$

for $\lambda \in \Sigma$ and $\gamma \in S^1$.

Factorization for pulses

$$u_\zeta = \begin{pmatrix} \varepsilon\mathcal{A}_{11,\varepsilon} & \varepsilon\mathcal{A}_{12,\varepsilon} \\ \mathcal{A}_{21,\varepsilon} & \mathcal{A}_{22,\varepsilon} \end{pmatrix}[\zeta,\lambda]\ u. \qquad \text{(LSP)}$$

1. Spectrum of asymptotic matrix $\mathcal{A}^\infty_\varepsilon(\lambda)$ of (LSP) is $\mathcal{O}(\varepsilon)$-close to

$$\{0\} \cup \sigma(\mathcal{A}^\infty_{22,\varepsilon}(\lambda)), \quad \text{with } |\mathrm{Re}\ \sigma(\mathcal{A}^\infty_{22,\varepsilon}(\lambda))| > \eta.$$

   where $\eta > 0$ is λ, ε-independent.
2. Split $U_\varepsilon(\lambda), S_\varepsilon(\lambda)$ according to spectral gap: $S_{s,\varepsilon}, S_{f,\varepsilon}, U_{s,\varepsilon}, U_{f,\varepsilon}$.
3. $\mathcal{T}_{d,\varepsilon}(\zeta,0)H_\varepsilon(0)S_{f,\varepsilon}$ decays faster than $e^{-\eta\zeta}$. Thus, $H_\varepsilon(0)S_{f,\varepsilon} = \binom{0}{*}$.

Factorization:

$$\begin{aligned} D_\varepsilon(\lambda) &= \det(S_{s,\varepsilon}|S_{f,\varepsilon}|U_{s,\varepsilon}|U_{f,\varepsilon})[\lambda] \\ &= \det(H_\varepsilon(0)S_{s,\varepsilon}|H_\varepsilon(0)S_{f,\varepsilon}|H_\varepsilon(0)U_{s,\varepsilon}|H_\varepsilon(0)U_{f,\varepsilon})[\lambda] \\ &= D_{s,\varepsilon}(\lambda)D_{f,\varepsilon}(\lambda) \end{aligned}$$

for $\lambda \in \Sigma$.

Two issues

Issue 1: how to obtain a λ, ε-uniform exponential dichotomy on $\mathbb{R}$ for

$$v_\zeta = \mathcal{A}_{22,\varepsilon}(\zeta, \lambda)v? \tag{FP}$$

Singular limit: reduced fast problem

By **normal hyperbolicty:** there exists $\mu > 0$ such that

$$v_\zeta = \mathcal{A}_{22,0}(\zeta, \lambda)v, \tag{RFP}$$

has only point spectrum σ_p on $\mathrm{Re}(\lambda) > -\mu$.

Take $\delta > 0$. **Roughness** yields desired dichotomy of (FP) for

$$\lambda \in \Sigma := \{\mathrm{Re}(\lambda) > -\mu\} \setminus \underbrace{\mathcal{N}(\sigma_p, \delta)}_{\delta\text{-neighborhood}} .$$

Note: $D_{f,0}(\lambda)$ **is** Evans function of (RFP).

Two issues

Issue 2: Factorization $D_\varepsilon(\lambda) = D_{f,\varepsilon}(\lambda)D_{s,\varepsilon}(\lambda)$ holds only on subset $\Sigma \subset \mathbb{C}$.

Use that both $D_\varepsilon(\lambda)$ and $D_0(\lambda) = D_{f,0}(\lambda)D_{s,0}(\lambda)$ are analytic on $\mathrm{Re}(\lambda) > -\mu$ and apply **winding number arguments**.

- Note **zero-pole cancellation**.
- Potential problem about **translational** eigenvalue $\lambda = 0$.

Goal 2: employ factorization method

Employ factorization method

Model: RD system with scale separation

$$\begin{cases} u_t &= D_1 u_{yy} - H(u,v,\varepsilon) \\ v_t &= \varepsilon^2 D_2 v_{yy} - G(u,v,\varepsilon)v \end{cases}, \qquad u(x,t) \in \mathbb{R}^m, v(x,t) \in \mathbb{R}^n$$

Allow for semi-strong interaction:

$$H(u,v,\varepsilon) = H_1(u,v,\varepsilon) + \varepsilon^{-1} H_2(u,v)v.$$

Motivation

- General class that includes the **Gierer-Meinhardt** equations.
- If $n = 1$ having $H_2 \not\equiv 0$ can prevent solutions from being **unstable**.

Rescaled system $y = \varepsilon x$

$$\begin{cases} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u,v,\varepsilon) - \varepsilon H_2(u,v)v \\ v_t &= D_2 v_{xx} - G(u,v,\varepsilon)v \end{cases}, \qquad \text{(PDE)}$$

Employ factorization method

Stationary solutions to (PDE) satisfy

$$\begin{cases} D_1 u_x &= \varepsilon p \\ p_x &= \varepsilon H_1(u,v,\varepsilon) + H_2(u,v)v \\ D_2 v_x &= q \\ q_x &= G(u,v,\varepsilon)v \end{cases} \tag{ODE}$$

Invariant space $v = q = 0$ yields **slow reduced dynamics**

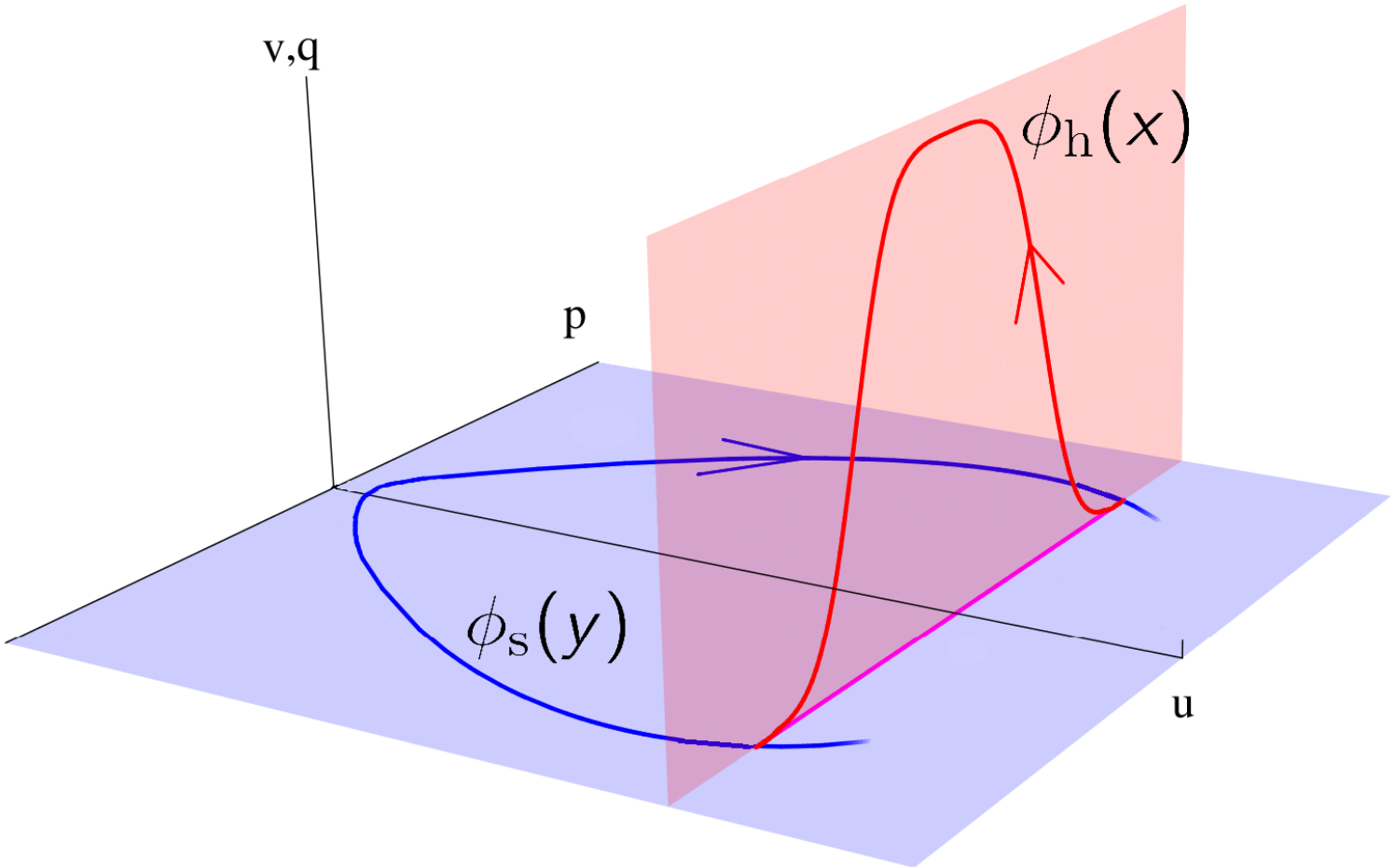
$$\begin{cases} D_1 u_y &= p \\ p_y &= H_1(u,0,0) \end{cases} \tag{SLOW}$$

Setting $\varepsilon = 0$ gives **fast layer problem**

$$\begin{cases} u_x &= 0 \\ p_x &= H_2(u,v)v \\ D_2 v_x &= q \\ q_x &= G(u,v,\varepsilon)v \end{cases} \tag{FAST}$$

Employ factorization method

Singular periodic orbit ϕ_{sing}: concatenation of orbits $\phi_{\mathrm{s}}(y)$ in (SLOW) and $\phi_{\mathrm{h}}(x)$ in (FAST).



Actual periodic solution $\phi_{\mathrm{p},\varepsilon}$ lies in ε-vicinity of ϕ_{sing}.

Employ factorization method

$$\begin{cases} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u,v,\varepsilon) - \varepsilon H_2(u,v)v \\ v_t &= D_2 v_{xx} - G(u,v,\varepsilon)v \end{cases}, \qquad \text{(PDE)}$$

Reduced fast problem

$$\psi_x = \mathcal{A}_{22,0}(x,\lambda)\psi, \quad \mathcal{A}_{22,0}(x,\lambda) = \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0, v_{\mathrm{h}}(x), 0) + \lambda & 0 \end{pmatrix}$$

with associated Evans function $D_{f,0}(\lambda)$ on $\mathrm{Re}(\lambda) > -\mu$.

Take $\delta > 0$. For each $\lambda \in \{\mathrm{Re}(\lambda) > -\mu\} \setminus \mathcal{N}(D_{f,0}^{-1}(0), \delta)$ we have

$$D_\varepsilon(\lambda,\gamma) = D_{f,\varepsilon}(\lambda,\gamma) D_{s,\varepsilon}(\lambda,\gamma), \quad 0 < \varepsilon \ll 1$$

Employ factorization method

Passage to **singular limit**

$$D_0(\lambda,\gamma) = D_{f,0}(\lambda) D_{s,0}(\lambda,\gamma).$$

Singular limit $D_{s,0}(\lambda,\gamma)$ is **explicit** in terms of

Inhomogeneous FRP

$V_{in}(x,\lambda)$ upper-left $n \times n$-block of **meromorphic** solution $\Phi(x,\lambda)$ to

$$\Phi_x = \mathcal{A}_{22,0}(x,\lambda)\Phi + \mathcal{A}_{21,0}(x), \quad \mathcal{A}_{21,0}(x) := \begin{pmatrix} 0 & 0 \\ \partial_u G(u_0, v_{\mathrm{h}}(x), 0) & 0 \end{pmatrix} \tag{IFRP}$$

Slow reduced problem

Evolution $\mathcal{T}_s(y,\tilde{y},\lambda)$ of

$$\begin{cases} D_1 u_y &= p \\ p_y &= (\partial_u H_1(u_{\mathrm{s}}(y),0,0) + \lambda)\, u \end{cases} \tag{SRP}$$

Employ factorization method

Passage to **singular limit**

$$D_0(\lambda,\gamma) = D_{f,0}(\lambda) D_{s,0}(\lambda,\gamma).$$

- $D_{f,0}(\lambda)$ **analytic** Evans function associated to (FRP).
- $D_{s,0}(\lambda)$ **meromorphic** function explicit in terms of (IFRP) and (SRP)

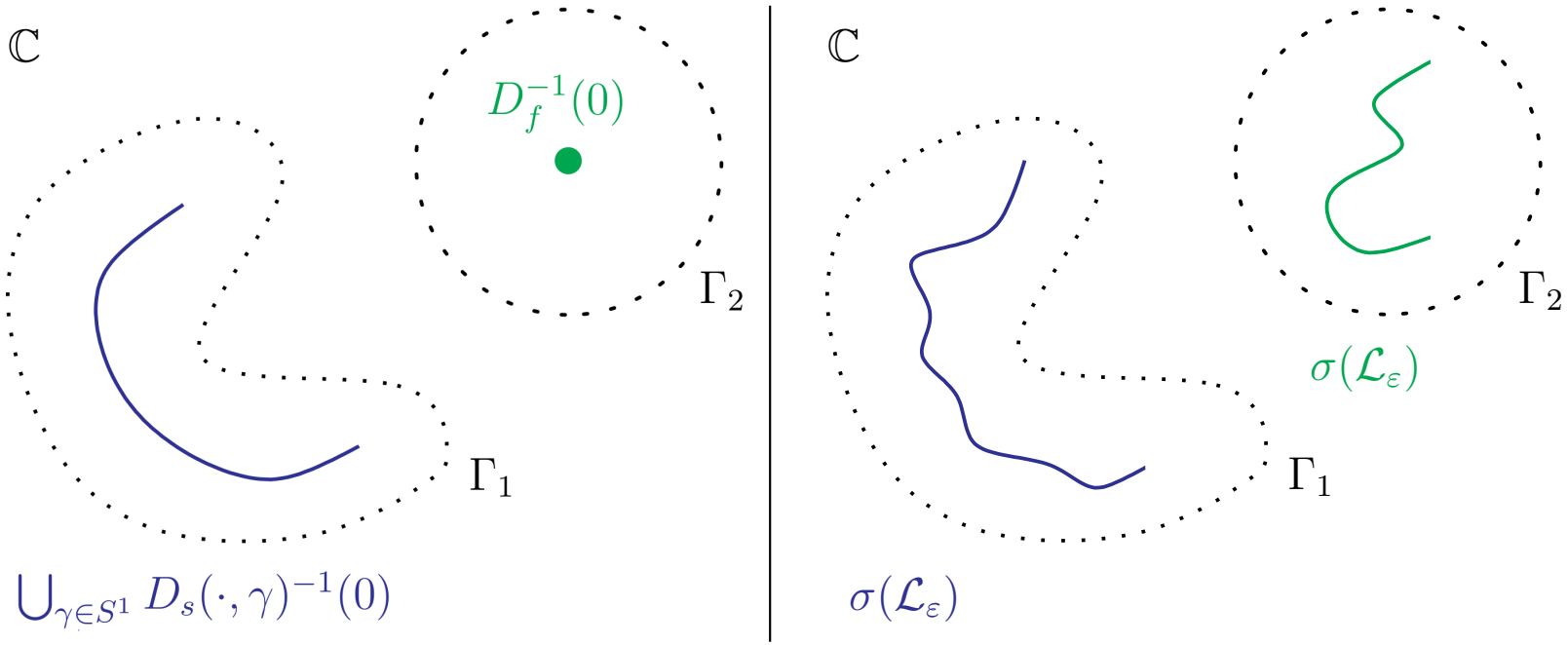
$$D_{s,0}(\lambda,\gamma) = (-\gamma)^n \det\left(\Upsilon(\lambda)\mathcal{T}_s(2L_0,0,\lambda) - \gamma I\right),$$

$$\Upsilon(\lambda) = \begin{pmatrix} 1 & 0 \\ \mathcal{G}(\lambda) & 1 \end{pmatrix},$$

$$\mathcal{G}(\lambda) = \int_{-\infty}^{\infty} [\partial_u H_2(u_0, v_{\mathrm{h}}(x)) + \partial_v H_2(u_0, v_{\mathrm{h}}(x)) V_{in}(x,\lambda)] dx.$$

Employ factorization method

Rouché's Theorem on difference $D_\varepsilon(\lambda,\gamma) - D_0(\lambda,\gamma)$ yields



approximation of spectrum $\sigma(\mathcal{L}_\varepsilon)$ (with ε-continuous error).

- **Reduction** of $2(n+m)$-dimensional, ε-dependent (LSP) to $2n$- and $2m$-dimensional problems (FRP), (IFRP) and (SRP).
- **Explicit** instability criteria (parity arguments).

Summary of results

- **Analytic** factorization method via **Riccati transform** formalizes and generalizes existing methods.
- Allows for passage to **singular limit** $D_0 = D_{f,0} D_{s,0}$, with $D_{f,0}$ analytic and $D_{s,0}$ meromorphic.
- **Explicit expressions** for $D_{f,0}$ and $D_{s,0}$ in the case of stationary wave trains in a broad, multi-dimensional class of RD-equations.

Future research

- Analysis of **spectrum** around 0. (Lyapunov-Schmidt reduction)
- Connection of wave trains to pulses in **long wavelength limit**. (Hopf dance)
- Introduce more **spatial dimensions**.

Questions

Thank you for the attention!

B. de Rijk, A. Doelman, J.D.M. Rademacher (2015). Spectra and stability of spatially periodic pulse patterns: Evans function factorization via Riccati transformation, *preprint on arXiv*