Nonlinear stability of fast invading fronts in a Ginzburg-Landau equation with an additional conservation law

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Abstract

We consider traveling front solutions connecting an invading state to an unstable ground state in a Ginzburg-Landau equation with an additional conservation law. This system appears as the generic amplitude equation for Turing pattern forming systems admitting a conservation law structure such as the Bénard-Marangoni problem. We prove the nonlinear stability of sufficiently fast fronts with respect to perturbations which are exponentially localized ahead of the front. The proof is based on the use of exponential weights ahead of the front to stabilize the ground state. The main challenges are the lack of a comparison principle and the fact that the invading state is only diffusively stable, i.e. perturbations of the invading state decay polynomially in time.

Keywords. Nonlinear stability; Invading fronts; Exponential weights
Mathematics Subject Classification (2020). 35B35; 35C07; 35Q56

1 Introduction

We consider the Ginzburg-Landau equation in the case of an additional conservation law

$$\partial_t A = \partial_x^2 A + A + AB - A|A|^2, \tag{1a}$$

$$\partial_t B = \mu \partial_x^2 B + \gamma \partial_x^2 (|A|^2), \tag{1b}$$

where $A(x,t) \in \mathbb{C}$, $B(x,t) \in \mathbb{R}$ and $\mu > 0$, $\gamma \in \mathbb{R}$. This model arises naturally as the generic amplitude equation for Turing pattern forming systems admitting a conservation law structure, see e.g. [MC00, HSZ11, SZ13, SZ17, Hil20] for examples. A prominent example for such a model is the Bénard-Marangoni problem modeling a free surface flow over a heated bottom, see Figure 2.

The system (1) has real traveling front solutions $(A,B)(x,t) = (A_{\text{front}}, B_{\text{front}})(x - ct)$ connecting an invading state $(A_f, B_f) = (1,0)$ to the origin, see Figure 1. The existence of these fronts with velocity $c \geq 2$ can be established for γ close to zero using perturbation arguments, see [Hil20, Section 4] for details. In particular, these fronts appear in the construction of modulating front solutions in a Swift-Hohenberg equation with an additional conservation law,

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 \alpha_0 u + uv - u^3,$$
$$\partial_t v = \partial_x^2 v + \gamma \partial_x^2 (u^2),$$

with γ close to zero, $\varepsilon > 0$ small and $\alpha_0 > 0$. These solutions model the invasion of the unstable homogeneous background by a periodic state and thus presents a mechanism for pattern formation, see Figure 3. Using center manifold reduction, it is proved that the amplitude, up to higher-order errors in ε, is given by the traveling front solutions of (1), see [Hil20]. Therefore, as a first step in the understanding of the stability of the modulating front solutions, we consider the stability of fronts in (1).

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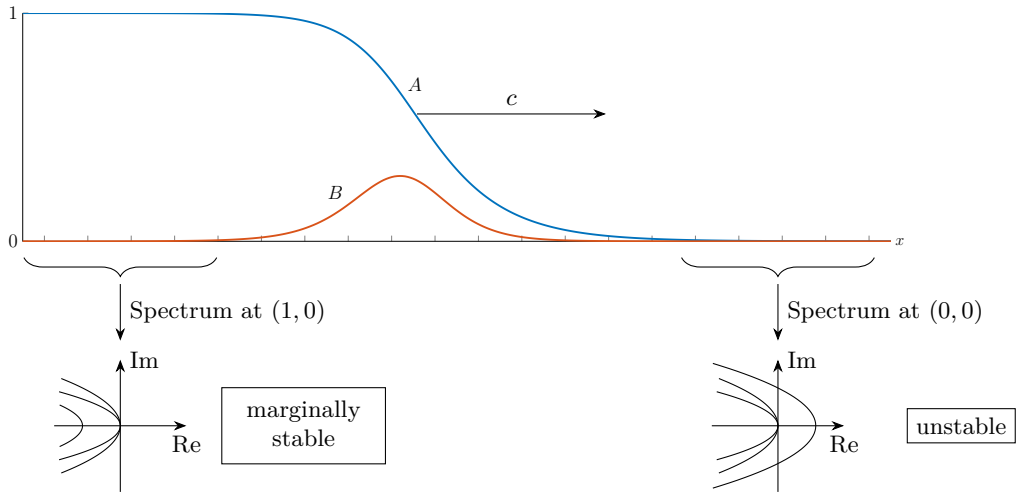

Figure 1: Plot of the front in (1) and spectral stability of rest states.

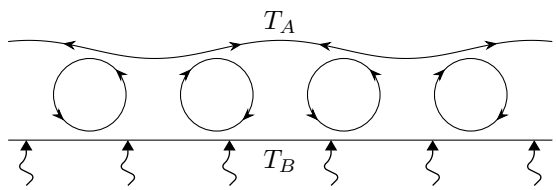

Figure 2: Schematic depiction of the Bénard-Marangoni convection close to the first instability, with $T_B > T_A$. Reprinted from [Hil20], with permission from Elsevier.

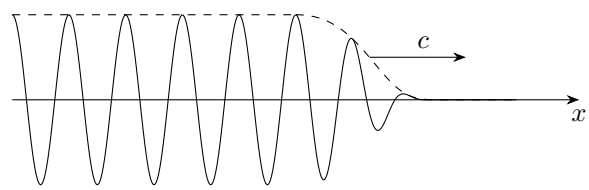

Figure 3: Modulating travelling front. Reprinted from [Hil20], with permission from Elsevier.

1.1 Main results and challenges

We now formulate the main results and discuss arising challenges. The set-up is as follows. Writing (1) in polar coordinates, i.e. $A = Re^{i\phi}$ and introducing the local wave number $\psi := \partial_x \phi$ we obtain

$$\partial_t R = \partial_x^2 R + R + RB - R^3 - R\psi^2, \tag{2a}$$

$$\partial_t B = \partial_x^2 B + \gamma \partial_x^2 (R^2), \tag{2b}$$

$$\partial_t \psi = \partial_x^2 \psi + 2\partial_x \frac{(\partial_x R)\psi}{R}. \tag{2c}$$

Since the front is real, it holds that $\psi_{\text{front}} = 0$ and the fronts connect the invading state $(A_f, B_f, \psi_f) = (1, 0, 0)$ to the origin. Linearising about the origin, we find essential spectrum with positive real part and thus, the origin is spectrally unstable, see Figure 1. In contrast, the linearisation about $(1, 0, 0)$ in Fourier space is given by

$$\hat{\tilde{L}}(k) = \begin{pmatrix} -k^2 - 2 & 1 & 0 \\ -2\gamma k^2 & -\mu k^2 & 0 \\ 0 & 0 & -k^2 \end{pmatrix},$$

$$\text{i.e. } \sigma(\hat{\tilde{L}}(k)) = \left\{ \frac{1}{2}\left(k^2(-(\mu+1)) \pm \sqrt{k^4(\mu-1)^2 - 4k^2(2\gamma+\mu-1)+4} - 2 \right), -k^2 \right\},$$

for $k \in \mathbb{R}$ and the L^2-spectrum of $\tilde{L}$ is given by the union of the of $\sigma(\hat{\tilde{L}}(k))$ over all $k \in \mathbb{R}$. In particular, if $\gamma > -\mu$, the state $(1,0,0)$ is spectrally stable with spectrum up to the imaginary axis as depicted in Figure 1.

1.1.1 Nonlinear stability of the invading state $(1,0,0)$

We start by establishing the nonlinear stability of the invading state $(1,0,0)$ with respect to localized perturbations. This is indeed a nontrivial task since the linearization has essential spectrum up to the imaginary axis. Let (r_A, r_B, r_ψ) be a perturbation of $(A_f, B_f, \psi_f) = (1,0,0)$, which satisfies

$$\begin{aligned}
\partial_t r_A &= \partial_x^2 r_A - 2r_A + r_B + r_A r_B - 3r_A^2 - r_A^3 - (1+r_A) r_\psi^2, \\
\partial_t r_B &= \mu \partial_x^2 r_B + \gamma \partial_x^2 (2r_A + r_A^2), \\
\partial_t r_\psi &= \partial_x^2 r_\psi + 2\partial_x \frac{(\partial_x r_A) r_\psi}{1+r_A}.
\end{aligned}$$

We rewrite this by introducing the dummy variable $r_C := \partial_x r_A$ and using $\partial_x^2(r_A^2) = 2\partial_x(r_A r_C)$. Then, we consider the extended system for $r = (r_A, r_C, r_B, r_\psi)$, which we abbreviate by

$$\partial_t r = Lr + \mathcal{N}(r). \tag{3}$$

Here, L denotes the linearisation about the invading state and $\mathcal{N}$ the nonlinearity. In particular, the system is locally well-posed in the Sobolev space H^m for $m > 1/2$ using standard methods, see e.g. [Hen81].

As expected, the L^2-spectrum of L is given by two curves touching the imaginary axis, while the remaining spectrum has a strictly negative real part, see Figure 6. Moreover, we find that L is diagonizable in Fourier space locally about $k = 0$, see Lemma 2.1. Thus, locally diagonalizing the system leads to two polynomially decaying variables and two exponentially decaying ones. We then use this separation to prove the nonlinear stability of the invading state if the perturbation is initially sufficiently small in a weighted Sobolev space H_2^m, see (11).

Theorem 1.1. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that for all perturbations (r_A, r_B, r_ψ) of the invading state $(A_f, B_f, \psi_f) = (1,0,0)$ satisfying*

$$\|(r_A, r_B, r_\psi)|_{t=0}\|_{H_2^{m+1} \times H_2^m \times (L^1 \cap H^m)} < \varepsilon$$

holds that

$$\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1} \times H^m \times H^m} \lesssim (1+t)^{-1/4}$$

for all $t \geq 0$.

Remark 1.2. To simplify the notation, in the above Theorem 1.1 and throughout the paper we use the notation $a \lesssim b$, i.e. there exists a constant $C < \infty$ such that $a \leq Cb$. □

Remark 1.3. We note that the use of polar coordinates is crucial in this stability analysis. It turns out that when using cartesean coordinates the analysis is much more subtile since the nonlinearities are of a less favorable form. This is to some extend not surprising since polar coordinates exploit the specific properties of the Ginzburg-Landau nonlinearity $A|A|^2$. □

Remark 1.4. Note that the difference in regularity, i.e. $r_A \in H^{m+1}$ and $r_B \in H^m$, originates in the introduction of the dummy variable $r_C = \partial_x r_A$. This reflects that the original system (1) is well-posed in $H^{m+1} \times H^m$ since it is semilinear in these spaces, see also [Zim14, Gau17]. □

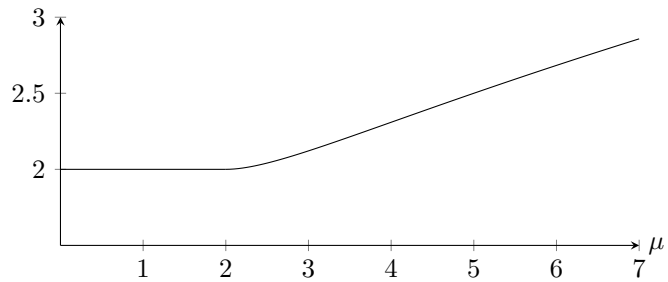

Figure 4: Plot of minimal velocity $c_{\min}(\mu)$, which is sufficient to prove nonlinear stability.

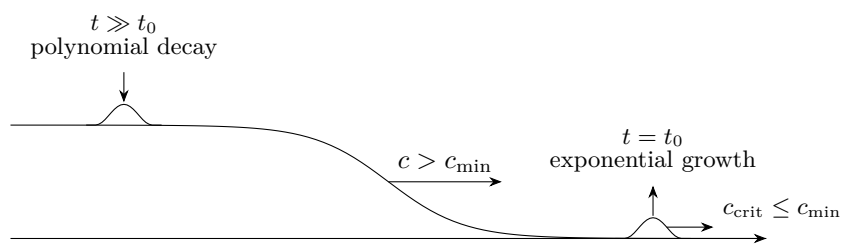

Figure 5: Expected behavior of small perturbations ahead of the bulk of the front. At time $t = 0$, the perturbation is located ahead of the front and grows exponentially. However, if the perturbation is small enough, the front reaches it for $t \gg 0$ and behind the bulk of the front the perturbation shows diffusive decay.

1.1.2 Nonlinear stability of the traveling front $(A_{\textbf{front}}, B_{\textbf{front}}, \psi_{\textbf{front}})$

We now outline the expected mechanism for the stability of sufficiently fast fronts. Here, a front is deemed sufficiently fast if its spreading speed c is strictly larger than

$$c_{\min}(\mu) = \begin{cases} 2, & 0 < \mu \leq 2, \\ \dfrac{\mu}{\sqrt{\mu - 1}}, & \mu > 2, \end{cases} \tag{4}$$

where μ is the diffusion parameter in (1), see Figure 4. In the scalar case such as the Fischer-KPP equation, we observe the following behavior, see Figure 5. For a sufficiently fast front, the front spreads faster than the perturbations. Therefore, if a perturbation is behind the front, it remains in this region and since the invading state is stable, the perturbation decays. Now, consider a localized perturbation ahead of the front. Since the origin is unstable, the perturbation grows exponentially fast in time as long as it is small. However, since the front is faster than the perturbation, the perturbation is transported to the stable region behind the front. If the perturbation is still sufficiently small at that point in time, it then decays. This idea for stability was first proposed by Sattinger [Sat76, Sat77] and then later used for example by Eckmann and Schneider [ES02]. It turns out that a similar mechanism also is applicable in our setting.

Next, we outline how to turn this idea into a mathematical proof. Let (r_A, r_B, r_ψ) be a perturbation of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$ and $r_C := \partial_x r_A$ as above. Then, $r = (r_A, r_C, r_B, r_\psi)$ satisfies

$$\partial_t r = L_{\text{front}} r + \mathcal{N}_2(r),$$

where $L_{\text{front}} = L_{\text{front}}(\xi)$, with $\xi = x - ct$, denotes the linear part and $\mathcal{N}_2$ the nonlinear part. We rewrite this as

$$\partial_t r = Lr + \mathcal{N}_1(r) + \mathcal{N}_2(r), \tag{5}$$

where L is the linearisation about the (extended) invading state $(1,0,0,0)$ and $\mathcal{N}_1 := L_{\text{front}} - L$. In particular, L is now a constant-coefficient operator, independent of (x,t). Furthermore, we highlight that $\mathcal{N}_1$ vanishes exponentially fast as $\xi \to -\infty$, since $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}}) \to (1,0,0)$ for $\xi \to -\infty$.

To use this behavior, we introduce an exponentially weighted variable $w_j(\xi,t) := r_j(x+ct,t)e^{\beta_j \xi}$ for $j \in \{A, C, B, \psi\}$ and some $\beta_j > 0$ to be determined later. This variable satisfies the weighted problem

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w,r), \tag{6}$$

with $w = (w_A, w_C, w_B, w_\psi)$, where $\mathcal{N}_\beta(w,r)$ is linear in w but nonlinear as a function of both w and r.

Remark 1.5. It turns out that we can choose $\beta_A = \beta_C = \beta_B$ in the weighted variable. Therefore, we only denote β_ψ seperately and otherwise write $\beta = \beta_j$, $j \in \{A, C, B\}$. □

By considering the coupled (r,w)-system (5)–(6), we can show that $\mathcal{N}_1$ can be treated as a nonlinear contribution in the stability analysis, which is already hinted by the notation. The major benefit of this strategy is that it is not necessary to control the spectrum of L_{front}. Instead, the difficulty shifts to the spectral analysis of the weighted operator L_β. However, it turns out that the spectrum of L_β has strictly negative real part for properly chosen $\beta, \beta_\psi > 0$ and γ close to zero, see the subsequent Theorem 1.6.

Although this idea has already been used by Sattinger [Sat76, Sat77] and Eckmann and Schneider [ES02], the implementation in these cases relied heavily on the use of comparison principle to show that w is linearly exponentially stable. However, in our setting, there is no comparison principle available, see Remark 1.11. Instead, we use that for $\gamma = 0$, for which the conservation law decouples, we can establish estimates for the spectrum of L_β. Then, considering γ close to zero as a small perturbation we establish spectral stability, see Lemma 3.1. Using this spectral stability we can prove the nonlinear stability of the invading fronts, cf. Theorem 3.9.

Theorem 1.6. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$, see (4). Then, there exists a $\beta_\psi^0 > 0$ such that for $\beta > 0$ and $\beta_\psi > 0$ satisfying*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta + c) < 0,\\
&\beta_\psi \in (0, \beta_\psi^0),\\
&\beta \le \frac{1}{2}(c - \sqrt{c^2 - 4}) + \beta_\psi,
\end{aligned}$$

exists a $\varepsilon > 0$, $\varepsilon_w > 0$, $\gamma_0 > 0$ and a $\bar{\kappa} > 0$ such that if

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)|_{t=0}\|_{H_2^{m+1} \times H_2^m \times (L^1 \cap H^m)} < \varepsilon\\
&\left\|(r_A, r_B)|_{t=0} e^{\beta x}\right\|_{H^{m+1} \times H^m} + \left\|r_\psi|_{t=0} e^{\beta_\psi x}\right\|_{H^m} < \varepsilon_w,
\end{aligned}$$

and $\gamma \in (-\gamma_0, \gamma_0)$ it holds that

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1} \times H^m \times H^m} \lesssim (1+t)^{-1/4},\\
&\left\|(r_A, r_B)(t) e^{\beta(x-ct)}\right\|_{H^{m+1} \times H^m} + \left\|r_\psi(t) e^{\beta_\psi(x-ct)}\right\|_{H^m} \lesssim e^{-\bar{\kappa} t}
\end{aligned}$$

for all $t \ge 0$.

Remark 1.7. Note that the existence of $\beta, \beta_\psi > 0$ in the above theorem is guaranteed assuming that $c > c_{\min}(\mu)$. However, we want to stress that $c_{\min}(\mu)$ is not necessarily the critical spreading speed of the problem, i.e. the speed with which steep enough inital data spreads. In fact the existence of such a critical spreading speed and its value, in particular for $\mu > 2$, is an open question. □

Remark 1.8. Since all fronts for $c > c_{\min}(\mu)$ are stable, a natural question is which speed is chosen given some initial data. A closer analysis of the weights yields that in order to obtain stability of a front with some speed $c > c_{\min}(\mu)$ we have to require that the perturbation decays faster than the front for $\xi \to \infty$. Therefore, a criterion for the selected speed is the decay rate of the initial data for $\xi \to \infty$. □

Remark 1.9. Recall that the existence of the invading fronts has only been established for γ close to zero. Therefore, the nonlinear stability result Theorem 1.6 covers a similar parameter regime for which the existence of fronts is known. However, it is important to note that this restriction hinges purely on the spectral estimates. In particular, given spectral stability we can show nonlinear stability also for large $|\gamma|$. □

Remark 1.10. In Theorem 1.6 we assume that the front is sufficiently fast, i.e. $c > c_{\min}(\mu)$. In the scalar case, e.g. $\partial_t u = \partial_x^2 u + u - u^3$, stability has also been proven for fronts with minimal velocity for which L_β has essential spectrum up to the imaginary axis, see e.g. [Kir92, Gal94, FH19a]. For systems, similar results have only been proven in special cases for which a comparison principle holds such as a Lotka-Volterra competition model [FH19b]. However, if the system lacks a comparison principle, as is for the system in this paper (see Remark 1.11), similar stability questions are still open, see Section 5 for a discussion. □

Remark 1.11. We note that even in the case of real perturbation, where the Ginzburg-Landau equation degenerates to a KPP equation, the system (1) lacks a comparison principle. For a counter example, note that $(0, B)$, $B \in \mathbb{R}$ is an invariant set of (1). Thus, for any $\gamma \neq 0$ such that a front $(A_{\text{front}}, B_{\text{front}})$ with some speed $c > 2$ exists, take initial data $(0, B_0)$ such that $|B_{\text{front}}| \geq |B_0|$. The solution to this initial data is explicitly given as a solution of the heat equation $\partial_t B = \mu \partial_x^2 B$. Hence, B decays polynomially to zero as time tends to infinity. However, since the front moves with speed $c > 2$ and $B_{\text{front}}(\xi)$ tends to zero exponentially fast as $\xi \to -\infty$ there exists $(t_0, x_0) \in \mathbb{R}^2$ such that $|B_{\text{front}}(x_0 - ct_0)| \leq |B(t_0, x_0)|$. Thus, the comparison principle cannot hold. □

1.2 Outline of the paper

The plan for this paper is as follows. In Section 2 we proof the nonlinear stability of the invading state $(1, 0, 0)$ of the system (2). Following, we establish nonlinear stability of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$, starting from a linear stability result Lemma 3.1. Section 4 then presents the proof of the aforementioned spectral and linear stability result. In Section 5 we discuss related, open problems and finally, we establish necessary properties of the invading fronts in Appendix A.

2 Nonlinear stability of the invading state

We now study the nonlinear stability of the invading state. A perturbation $(r_A, r_B, r_\psi) := (A, B, \psi) - (1, 0, 0)$ satisfies the equation

$$\partial_t r_A = \partial_x^2 r_A - 2r_A + r_B + r_A r_B - 3r_A^2 - r_A^3 - (1 + r_A) r_\psi^2, \tag{7a}$$

$$\partial_t r_B = \mu \partial_x^2 r_B + \gamma \partial_x^2 (2r_A + r_A^2), \tag{7b}$$

$$\partial_t r_\psi = \partial_x^2 r_\psi + 2\partial_x \left(\frac{r_C r_\psi}{1 + r_A} \right). \tag{7c}$$

Note that this system is ill-posed when considered in $H^m \times H^m$ since the nonlinearity contains a second derivative. However, we can use the special structure of the nonlinearity by considering the system in $H^{m+1} \times H^m$, see also [Zim14, Gau17]. In particular, in this setting, the system is semilinear and thus locally well-posed by standard methods.

More specifically, in the subsequent analysis we exploit this structure by introducing a dummy variable $r_C := \partial_x r_A$ and use

$$\partial_x^2 (2r_A + r_A^2) = 2\partial_x (r_C + r_A r_C)$$

in order to turn (7) into a semilinear system in H^m. By differentiating (7a), r_C satisfies

$$\partial_t r_C = \partial_x^2 r_C - 2\alpha r_C - 2(1 - \alpha)\partial_x r_A + \partial_x (r_B + r_A r_B - 3r_A^2 - r_A^3 - (1 + r_A) r_\psi). \tag{8}$$

We abbreviate the four dimensional system (7)–(8) as

$$\partial_t r = Lr + \mathcal{N}(r), \tag{9}$$

with $r = (r_A, r_C, r_B, r_\psi)^T \in \{r \in \mathbb{R}^4 \,|\, \partial_x r_A = r_C\}$. Here L denotes the linear part and $\mathcal{N}$ the nonlinearities. In Fourier space, the system reads as

$$\begin{aligned}\partial_T \hat{r} &= \begin{pmatrix} -k^2-2 & 0 & 1 & 0 \\ -2(1-\alpha)ik & -k^2-2\alpha & ik & 0 \\ 0 & 2ik & -\mu k^2 & 0 \\ 0 & 0 & 0 & -k^2 \end{pmatrix} \hat{r} + \begin{pmatrix} \hat{r}_A * \hat{r}_B - 3\hat{r}_A^{*2} - \hat{r}_A^{*3} - (1+\hat{r}_A) * \hat{r}_\psi^{*2} \\ ik(\hat{r}_A * \hat{r}_B - 3\hat{r}_A^{*2} - \hat{r}^{*3} - (1+\hat{r}_A) * \hat{r}_\psi^{*2}) \\ 2\gamma ik(\hat{r}_A * \hat{r}_C) \\ 2ik\mathcal{F}(r_C r_\psi (1+r_A)^{-1}) \end{pmatrix} \\ &=: \hat{L}\hat{r} + \hat{\mathcal{N}}(\hat{r}).\end{aligned}$$

Here $\hat{(\cdot)}$ and $\mathcal{F}(\cdot)$ denote Fourier transform and $f * g$ denotes the convolution of the functions f and g. Additionally f^{*q} denotes q-times convolution of f with itself. Finally, we note that the $\hat{r}_\psi$ is linearly decoupled from the rest and thus, we write

$$\hat{L} = \begin{pmatrix} \hat{\tilde{L}} & 0 \\ 0 & -k^2 \end{pmatrix}.$$

2.1 Linear stability analysis

We begin the analysis by studying the linearized system. Therefore, we calculate the spectrum of L. Since one eigenvalue of $\hat{\tilde{L}}$ is trivially given by $\lambda_\psi(k) = -k^2$, we focus on the spectrum of $\tilde{L}$, which is given by

$$\begin{aligned}\sigma(\tilde{L}) = \bigcup_{k\in\mathbb{R}} \Bigg\{ &-\frac{1}{2}(\mu+1)k^2 - \frac{1}{2}\sqrt{(\mu-1)^2k^4 - 4(2\gamma+\mu-1)k^2+4} - 1, -k^2-2\alpha, \\ &\frac{1}{2}\left(-(\mu+1)k^2 + \sqrt{(\mu-1)^2k^4 - 4(2\gamma+\mu-1)k^2+4} - 2\right)\Bigg\} \\ = \bigcup_{k\in\mathbb{R}} &\{\lambda_A(k), \lambda_C(k), \lambda_B(k)\}.\end{aligned}$$

Note that for $\gamma > -\mu$ there exists a $\theta_j > 0, j \in \{A, C, B\}$ such that

$$\begin{aligned}\mathrm{Re}(\lambda_A(k)) &\le -\theta_A k^2 - 2, \\ \mathrm{Re}(\lambda_C(k)) &\le -\theta_C k^2 - 2\alpha, \\ \mathrm{Re}(\lambda_B(k)) &\le -\theta_B k^2\end{aligned}$$

see Figure 6. Thus, the system has two eigenspaces which correspond to exponential decay in time and two which correspond to polynomial decay in time. To extract this behavior we diagonalize the system in Fourier space using the following result.

Lemma 2.1. *Let $\mu > 0$, $\gamma > -\mu$ and $\alpha \in (0,1)$. Then there exists a $k_0 > 0$ such that the eigenvalue curves of $\hat{\tilde{L}}(k)$ do not intersect for any $k \in (-k_0, k_0)$ and thus, $\hat{\tilde{L}}(k)$ is diagonizable via an invertible transformation $\hat{\tilde{S}}(k)$, i.e.*

$$\hat{\tilde{\Lambda}}(k) := \mathrm{diag}(\lambda_A(k), \lambda_C(k), \lambda_B(k)) = \hat{\tilde{S}}(k)^{-1}\hat{\tilde{L}}(k)\hat{\tilde{S}}(k)$$

for all $k \in (-k_0, k_0)$. Furthermore, the transformation can be chosen such that, locally about $k = 0$, the following element-wise estimates hold

$$|\hat{\tilde{S}}(k)| \lesssim \begin{pmatrix} 1 & |k| & 1 \\ |k| & 1 & |k| \\ k^2 & |k| & 1 \end{pmatrix} \text{ and } |\hat{\tilde{S}}^{-1}(k)| \lesssim \begin{pmatrix} 1 & |k| & 1 \\ |k| & 1 & 0 \\ k^2 & |k| & 1 \end{pmatrix}.$$

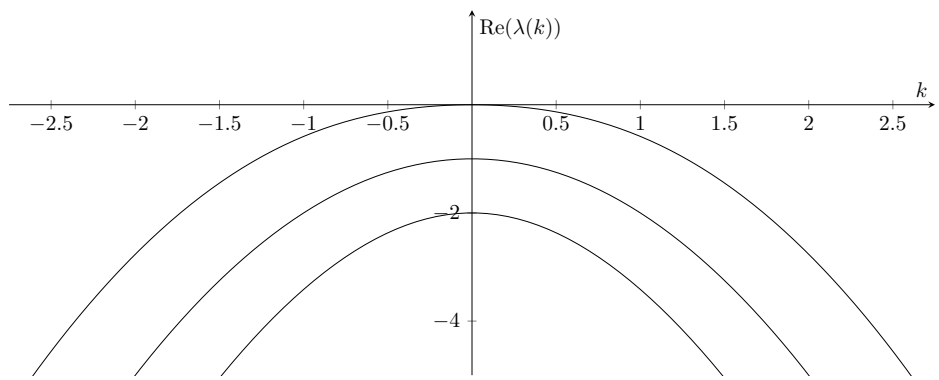

Figure 6: Eigenvalue curves of $\tilde{L}$ for $\alpha = 1/2$, $\gamma = -1/2$, $\mu = 1$, which are parameterized by Fourier wave number k.

Here, both the inequality as well as the absolute value are defined by element-wise application.

Proof. Using the given formulas for λ_j, $j \in \{A, C, B\}$ it is straightforward to check that there is no intersection at $k = 0$. Thus, by continuity of eigenvalues there exists a $k_0 > 0$ such that $\hat{L}(k)$ is diagonizable for all $k \in (-k_0, k_0)$. Finally, we verify the estimates by explicitly calculating the eigenvectors and noting that these are only unique up to normalization. □

Remark 2.2. Note that the parameter $\alpha \in (0, 1)$ separates the eigenvalue curves λ_A and λ_C. □

Motivated by the above result, we introduce a smooth and symmetric cut-off function $\chi_{k_0} : \mathbb{R} \to [0, 1]$ such that $\chi_{k_0}(k) = 1$ if $k \in (-k_0/2, k_0/2)$ and $\chi_{k_0}(k) = 0$ if $k \notin (-k_0, k_0)$. Then, we define

$$\hat{S}(k) := \chi_{k_0} \begin{pmatrix} \hat{\tilde{S}}(k) & 0 \\ 0 & 1 \end{pmatrix} + (1 - \chi_{k_0})I, \tag{10}$$

where I is the 4×4 identity matrix. Furthermore, since $\hat{\tilde{S}}(k = 0) + I$ is invertible (by explicit calculation of $\hat{\tilde{S}}(k = 0)$), $\hat{S}(k)$ is invertible for every $k \in \mathbb{R}$ if $|k_0|$ is chosen small enough. Therefore, we define $\hat{S}^{-1}(k)$ as the inverse of $\hat{S}(k)$ for every $k \in \mathbb{R}$. Since for k close to zero, $\chi_{k_0} = 1$, this transformation diagonalizes $\hat{L}(k)$ locally around $k = 0$. Thus, it extracts the different linear behavior from the system. Next, we introduce $\hat{p} = (\hat{p}_A, \hat{p}_C, \hat{p}_B, \hat{p}_\psi) := \hat{S}^{-1}\hat{r}$, which then satisfies $\partial_T \hat{p} = \hat{\Lambda}\hat{p} + \hat{S}^{-1}\hat{\mathcal{N}}(\hat{S}\hat{p})$, with $\hat{\Lambda} := \hat{S}^{-1}\hat{L}\hat{S}$. Furthermore, we expect that $\hat{p}_A$ and $\hat{p}_C$ decay exponentially fast on the linear level, while $\hat{p}_B$ and $\hat{p}_\psi$ show polynomial decay, since $\lambda_B(k = 0) = \lambda_\psi(k = 0) = 0$.

Remark 2.3. Note that $p_\psi = r_\psi$ and in particular, p_ψ is does not depend on r_A, r_C and r_B via the diagonalisation. Vice versa, p_A, p_C, p_B are determind by a linear combination of r_A, r_C, r_B. □

2.2 Function spaces and semigroup estimates

Starting from the operators in $\hat{\Lambda}, \hat{S}$ and $\hat{S}^{-1}$, which operate in Fourier space, we define the corresponding operators in the physical space as Fourier multipliers, e.g. $S = \mathcal{F}^{-1}\hat{S}\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. Therefore, the Sobolev spaces H^m (see e.g. [SU17]) are a natural setting since H^m is isomorphic to $\widehat{H_m}$ via Fourier transform. Here $\widehat{H_m}$ is a weighted L^2 space defined by

$$\widehat{H_m} := \left\{ u \in L^2 : \|\hat{\rho}(k)^m u(k)\|_{L^2(k)} < \infty \right\},$$

where $\hat{\rho}(k) := (1+k^2)^{1/2}$, see [SU17]. Furthermore, we note that for $m > 1/2$, we have

$$\|fg\|_{H^m} \lesssim \|f\|_{H^m} \|g\|_{H^m} ,$$
$$\left\|\hat{f} * \hat{g}\right\|_{\widehat{H_m}} \lesssim \left\|\hat{f}\right\|_{\widehat{H_m}} \|\hat{g}\|_{\widehat{H_m}} ,$$

that is, H^m and $\widehat{H_m}$ are Banach algebras with respect to pointwise multiplication and convolution, respectively. Additionally, we introduce the weighted Sobolev spaces H^m_n as

$$H^m_n := \{u \in L^2 : \left\|(1+x^2)^{n/2} u\right\|_{H^m} < \infty\}, \tag{11}$$

where we usually set $n = 2$ in what follows. Similar to the classical Sobolev space H^m, we note that the Fourier transform is an isomorphism from H^m_n to H^n_m.

Now, let P_j, $j \in \{A, C, B, \psi\}$ be the projection on the j-th component. Then, we can prove the following linear stability estimates.

Corollary 2.4. *Let $\mu > 0$, $\gamma > -\mu$, $m \in \mathbb{N}_0$ and $\alpha \in (0,1)$. Furthermore, let*

$$|\hat{\rho}_{n_1,n_2}(k)| \lesssim |k|^{n_1} (|k|/\sqrt{1+k^2})^{n_2}$$

for $n_1, n_2 \in \mathbb{N}_0$. Then, there exists a $\theta > 0$ such that for any $p = (p_A, p_C, p_B, p_\psi) \in H^m$ holds that

$$\begin{aligned}
\left\|P_A e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim t^{-n_1/2} e^{-2t} \|p_A\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m} ,\\
\left\|P_C e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim t^{-n_1/2} e^{-2\alpha t} \|p_C\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m} ,\\
\left\|P_B e^{t\Lambda} p\right\|_{H^m} &\lesssim (1+t)^{-1/4} \|p_B\|_{L^1 \cap H^m} + e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m} ,\\
\left\|P_B e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim (1+t)^{-n_2/2} t^{-n_1/2} \|p_B\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m} ,\\
\left\|P_\psi e^{t\Lambda} p\right\|_{H^m} &\lesssim (1+t)^{-1/4} \|p_\psi\|_{L^1 \cap H^m} ,\\
\left\|P_\psi e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim (1+t)^{-n_2/2} t^{-n_1/2} \|p_\psi\|_{H^m} .
\end{aligned}$$

Proof. We use the isomorphism of H^m and $\widehat{H_m}$ and the construction of the transformation S, see (10). Let $j \in \{A, C, B\}$. Then for some $\theta > 0$ holds

$$\left\|P_j e^{t\Lambda} p\right\|_{H^m} \lesssim \left\|P_j e^{t\hat{\Lambda}} p\right\|_{\widehat{H_m}} \lesssim \left\|\chi_{k_0/2} e^{\lambda_j t} \hat{p}_j\right\|_{\widehat{H_m}} + e^{-\theta t} \|(\hat{p}_A, \hat{p}_C, \hat{p}_B)\|_{\widehat{H_m}} ,$$

since all eigenvalues of $\hat{\Lambda}(k)$ have strictly negative real part and are uniformly bounded away from the imaginary axis for $k \notin (-k_0/4, k_0/4)$. Furthermore, the first term separates the different behavior of the eigenvalues about $k = 0$. Thus, for $j = A, C$ we obtain the exponential bounds as states in the corollary by noting that $\mathrm{Re}(\lambda_A) \leq -2$ and $\mathrm{Re}(\lambda_C) \leq -2\alpha$, respectively. Finally, for $j = B$ we can calculate

$$\begin{aligned}
\left\|\chi_{k_0/2} e^{\lambda_B t} \hat{p}_B\right\|_{\widehat{H_m}} &\lesssim \left\|e^{-\theta_B k^2 t} \hat{p}_B\right\|_{\widehat{H_m}} \lesssim \min\left(\left\|e^{-\theta_B k^2 t}\right\|_{L^\infty} \|\hat{p}_B\|_{\widehat{H_m}} , \left\|e^{-\theta_B k^2 t}\right\|_{\widehat{H_m}} \|\hat{p}_B\|_{L^\infty}\right)\\
&\lesssim (1+t)^{-1/4} \|p_B\|_{L^1 \cap H^m} .
\end{aligned}$$

The estimates in case that $n_1, n_2 \geq 1$ can be obtain in a similar manner, using the behavior of λ_j locally about $k = 0$ and that the eigenvalues behave like $-k^2$ for $k \to \pm\infty$. Finally, since $P_\psi e^{t\Lambda} p = e^{t\partial_x^2} p_\psi$, the remaining estimates follow with the same techniques. □

2.3 Nonlinear stability

We now show nonlinear stability of the transformed variable $p := S^{-1} r$ satisfying the equation

$$\partial_t p = \Lambda p + S^{-1} \mathcal{N}(Sp). \tag{12}$$

That is, we prove the following result.

Lemma 2.5. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that all perturbations p of $(1,0,0,0)$ with*

$$\|(p_A, p_C, p_B, p_\psi)|_{t=0}\|_{H^m \times H^m \times (L^1 \cap H^m) \times (L^1 \cap H^m)} < \varepsilon$$

it holds that

$$\|p_A(t)\|_{H^m}, \|p_C(t)\|_{H^m} \lesssim (1+t)^{-1/2} \text{ and } \|p_B(t)\|_{H^m}, \|p_\psi(t)\|_{H^m} \lesssim (1+t)^{-1/4}. \tag{13}$$

From this result we can deduce the nonlinear stability of the nontrivial fixed point of (7), see Theorem 1.1, which we restate here.

Theorem 2.6. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that for all perturbations (r_A, r_B, r_ψ) of the invading state $(1,0,0)$ satisfying*

$$\|(r_A, r_B, r_\psi)|_{t=0}\|_{H_2^{m+1} \times H_2^m \times (L^1 \cap H^m)} < \varepsilon$$

holds that

$$\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1} \times H^m \times H^m} \lesssim (1+t)^{-1/4}$$

for all $t \geq 0$.

Proof. We define $r_0 = (r_A, \partial_x r_A, r_B)|_{t=0}$ and $(\hat{p}_0, 0) := \hat{S}^{-1}(\hat{r}_0, 0)$. Furthermore, we use that the Fourier transform is an isomorphism between H^m and $\widehat{H_m}$ and obtain

$$\|p_0\|_{H^m} \lesssim \|\hat{p}_0\|_{\widehat{H_m}} \lesssim \left\|\hat{S}^{-1}(\hat{r}_0, 0)\right\|_{\widehat{H_m}} \lesssim \|\hat{r}_0\|_{\widehat{H_m}} \lesssim \|r_0\|_{H^m} \lesssim \|(r_A, r_B)|_{t=0}\|_{H^{m+1} \times H^m},$$

with constants independent of r_0. We now bound the L^1-norm of p_0 by using that H^m is continuously embedded into L^∞ for $m > 1/2$ by Sobolev embedding. Then, it holds

$$\|p_0\|_{L^1} \lesssim \left\|(1+x^2)p_0\right\|_{L^\infty} \lesssim \|p_0\|_{H_2^m} \lesssim \|\hat{p}_0\|_{H_m^2} \lesssim \left\|\hat{S}^{-1}\right\|_{C_b^2} \|\hat{r}_0\|_{H_m^2} \lesssim \|r_0\|_{H_2^m} \lesssim \|(r_{A,0}, r_{B,0})\|_{H_2^{m+1} \times H_2^m},$$

where we used that every element of $\hat{S}^{-1}$ is in C_b^2 (see Lemma 2.1). Finally, we recall that $p_\psi = r_\psi$. Hence, for ε sufficiently small the conditions of Lemma 2.5 are satisfied. Furthermore, we obtain with $r_C := \partial_x r_A$ that

$$\|(r_A(t), r_C(t), r_B(t), r_\psi(t))\|_{H^m} \lesssim \left\|\hat{S}(\hat{p}_A(t), \hat{p}_C(t), \hat{p}_B(t), \hat{p}_\psi(t))\right\|_{\widehat{H_m}} \lesssim \|(p_A(t), p_C(t), p_B(t), p_\psi(t))\|_{H^m}$$

where the latter term decays to zero like $(1+t)^{-1/4}$. In particular, since we control r_A as well as $r_C = \partial_x r_A$ in H^m, we obtain decay of r_A in H^{m+1} which proves the statement. □

Remark 2.7. Since H^m is continuously embedded into L^∞ for $m > 1/2$ by Sobolev embedding, we obtain from Theorem 2.6 that the invading state is also $(H_2^{m+1} \times H_2^m \times (L^1 \cap H^m), L^\infty)$ asymptotically stable. However, we expect that the perturbation exhibits faster decay in L^∞, similar to solutions of the heat equation. □

Remark 2.8. We highlight that the stability result requires that $r_A|_{t=0} \in H_2^{m+1}$ and $r_B|_{t=0} \in H_2^m$, i.e., $r_A|_{t=0}$ is of higher regularity than $r_B|_{t=0}$. As discussed above this meshes well with the results on existence and uniqueness for the modified Ginzburg-Landau system (1) in [Gau17, Zim14]. □

Remark 2.9. Ideally it should be sufficient to assume smallness of the initial data in $(L^1 \cap H^{m+1}) \times (L^1 \cap H^m) \times (L^1 \cap H^m)$ in Theorem 2.6, since it is sufficient to require smallness in $H^m \times H^m \times (L^1 \cap H^m) \times (L^1 \cap H^m)$ in Lemma 2.5. However, it is unclear whether the transformation S^{-1} has sufficiently good mapping properties to prove this, e.g. if S^{-1} is an L^1-Fourier multiplier. To get around this problem, we assume that the initial data is small in the (stronger) weighted Sobolev space $H_2^{m+1} \times H_2^m \times (L^1 \cap H^m)$ in Theorem 2.6. □

We now outline the plan to prove Lemma 2.5. Recall that $\hat{L}(k)$ has four eigenvalue curves $\lambda_A(k)$, $\lambda_C(k)$, $\lambda_B(k)$ and $\lambda_\psi(k)$. Furthermore, recall that only the spectral curves λ_B and λ_ψ touch the imaginary axis, while both λ_A and λ_C are bounded away from the imaginary axis. Therefore, we expect that the main challenge in the stability proof comes from the stability of p_B and p_ψ, the linearly polynomially decaying variables. However, a closer inspection of the nonlinearity in (12) reveals that

$$P_B\hat{S}^{-1}\hat{\mathcal{N}}(\hat{S}\hat{p}) = \mathcal{O}(|k|) \text{ and } P_\psi\hat{S}^{-1}\hat{\mathcal{N}}(\hat{S}\hat{p}) = \mathcal{O}(|k|) \text{ for } k \to 0,$$

where we used Lemma 2.1. Since $k = 0$ is the Fourier mode with no decay – the eigenvalue curve touches the imaginary axis for $k = 0$ – the influence of the critical modes in a neighborhood of $k = 0$ is small and thus, stability is more likely. This can also be observed in the example $\partial_t u = \partial_x^2 u - \partial_x^n(u^2)$, where the origin is unstable for $n = 0$ but stable for $n = 1$ (see e.g. [Wei81, SU17]).

Next, we recall that a solution of (12) is given by the Duhamel formula

$$p(t) = e^{t\Lambda}p(t=0) + \int_0^t e^{(t-s)\Lambda}S^{-1}\mathcal{N}(Sp(s))\,ds.$$

Our goal is to show that

$$\eta_A(t) := \sup_{0\le s\le t}(1+s)^{1/2}\|p_A(s)\|_{H^m}, \tag{14a}$$

$$\eta_C(t) := \sup_{0\le s\le t}(1+s)^{1/2}\|p_C(s)\|_{H^m}, \tag{14b}$$

$$\eta_B(t) := \sup_{0\le s\le t}(1+s)^{1/4}\|p_B(s)\|_{H^m}, \tag{14c}$$

$$\widetilde{\eta_B}(t) = \sup_{0\le s\le t}(1+s)^{1/2}\|S_{CB}p_B(s)\|_{H^m}, \tag{14d}$$

$$\eta_\psi(t) := \sup_{0\le s\le t}(1+s)^{1/4}\|p_\psi(s)\|_{H^m}, \tag{14e}$$

are bounded uniformly for all time, which in turn proves Lemma 2.5. To be precise, we prove that the energy $E(t) := \eta_A(t) + \eta_C(t) + \eta_B(t) + \widetilde{\eta_B}(t) + \eta_\psi(t)$ is uniformly bounded, provided the initial data is small in a suitable function space. The main result to show this is the following estimate.

Lemma 2.10. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. Then, if*

$$\|(p_A, p_C, p_B, p_\psi)|_{t=0}\|_{H^m\times H^m\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon$$

for a small $\varepsilon > 0$, there exists a constant $K < \infty$ independent of ε such that

$$E(t) \le K(\varepsilon + E(t)^2) \tag{15}$$

for all $t \ge 0$ such that the solution $(p_A, p_C, p_B, p_\psi)(t)$ of (12) exists in H^m and is sufficiently small.

Proof. First, we note that the integral terms containing an exponentially decaying function in time are harmless, since the nonlinearity contains at most one derivative, the transformation $\hat{S}^{-1}$ is uniformly bounded from $\widehat{H_m}$ to $\widehat{H_m}$ and the eigenvalues of $\hat{L}$ decay like $-k^2$ for $k \to \pm\infty$. Furthermore, we have that the estimate

$$\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m} + \|p_B(s)\|_{H^m} \lesssim (1+s)^{-1/4}E(t)$$

holds for any $0 \le s \le t$. Thus, we focus on estimating p_B, $S_{CB}p_B$ and p_ψ. Using Lemma 2.1 and Corollary 2.4 we obtain for p_B that

$$\|p_B(t)\|_{H^m} \lesssim (1+t)^{-1/4}\|p_B|_{t=0}\|_{L^1\cap H^m} + e^{-\theta t}\|(p_A, p_C, p_B)|_{t=0}\|_{H^m} + \int_0^t \left\|P_B e^{(t-s)\Lambda}S^{-1}\mathcal{N}(Sp(s))\right\|_{H^m} ds.$$

To estimate the integral contribution we need the following estimate,

$$|(\hat{S}_{AB}\hat{p}_B * \hat{S}_C B\hat{p}_B)(k)| \lesssim |k||\hat{p}_B|^{*2}(k)$$

for all $k \in (-k_0/2, k_0/2)$, which is proven in the subsequent Proposition 2.11. We then use this estimate, the behavior of S^{-1} close to $k = 0$ (see Lemma 2.1) and Corollary 2.4 to obtain

$$\begin{aligned}
&\int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1}\mathcal{N}(Sp(s)) \right\|_{H^m} ds \\
&\lesssim \int_0^t e^{-\theta(t-s)} \max(1, (t-s)^{-1/2}) \left\|(p_A, p_C, p_B)(s)\right\|_{H^m}^2 ds \\
&\quad + \int_0^t ((1+t-s)^{-1} + (1+t-s)^{-1/2}(t-s)^{-1/2}) \left\|(p_A, p_C, p_B, p_\psi)(s)\right\|_{H^m}^2 ds \\
&\quad + 2|\gamma| \int_0^t (t-s)^{-1/2} (\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m})^2 ds \\
&\quad + 2|\gamma| \int_0^t (t-s)^{-1/2} \left(\|p_B(s)\|_{H^m} \left(\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m}\right)\right) ds \\
&\quad + 2|\gamma| \int_0^t (t-s)^{-1/2}(1+t-s)^{-1/2} \|p_B(s)\|_{H^m}^2 ds \\
&\lesssim (1+t)^{-1/4} E(t)^2,
\end{aligned}$$

as long as $p(t)$ is small in H^m. Therefore, we obtain

$$\eta_B(t) \lesssim \varepsilon + E(t)^2$$

for all $t \geq 0$ such that $p(t)$ is small in H^m. Similarly, we obtain the estimate for $S_{CB}p_B$ using that $\hat{S}_{CB}(k) = \mathcal{O}(|k|)$ for $k \to 0$ and thus, this gives an additional $(1+t-s)^{-1/2}$ using Corollary 2.4. To estimate p_ψ we note that since $m > 1/2$, C_b is continuously embedded into H^m and thus, since the H^m-norm of p is small, we can bound $1 + (Sp)_A$ away from zero uniformly in space. This yields

$$\begin{aligned}
\|p_\psi(t)\|_{H^m} &\lesssim (1+t)^{-1/4} \|p_\psi|_{t=0}\|_{L^1\cap H^m} + \int_0^t \left\| e^{-k^2(t-s)} k\mathcal{F}((Sp)_C(s)p_\psi(s)(1+(Sp)_A(s))^{-1}) \right\|_{\widehat{H_m}} ds \\
&\lesssim (1+t)^{-1/4} \|p_\psi|_{t=0}\|_{L^1\cap H^m} \\
&\quad + \int_0^t (t-s)^{-1/2} \|p_\psi(s)\|_{H^m} \left(\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m} + \|S_{CB}p_B(s)\|_{H^m}\right) ds \\
&\lesssim (1+t)^{-1/4}(\|p_\psi|_{t=0}\|_{L^1\cap H^m} + E(t)^2),
\end{aligned}$$

as long as $p(t)$ is small in H^m. Here, we used that $S_{CB}p_B$ shows better decay rates than p_B, i.e. $(1+t)^{-1/2}$ compared to $(1+t)^{-1/4}$. Note that this improved decay rate originates from the fact that $\hat{S}_{CB}(k) = \mathcal{O}(|k|)$ locally about $k = 0$.

Finally, we can establish estimates on η_A, η_C using Corollary 2.4, the fact that the problem is semilinear and thus, all appearing integrals are finite and that

$$\int_0^t e^{-\kappa(t-s)}(t-s)^{-n_1}(1+t-s)^{-n_2}(1+s)^{-1/2} ds \lesssim (1+t)^{-1/2}$$

for all $n_1 = 0, 1$ and $n_2 = 0, 1, 2$ and $\kappa > 0$. This proves the conjectured estimate. □

We now prove the proposition used in the above proof.

Proposition 2.11. *Let $\mu > 0$ and $\gamma > -\mu$. Then, for all $k \in (-k_0/2, k_0/2)$ holds that*

$$|(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k)| \lesssim |k||\hat{p}_B|^{*2}(k).$$

Proof. First, fix $\alpha \in (0,1)$ such that Lemma 2.1 applies. To study the behavior about $k=0$, we note that $\hat{\hat{S}}_{CB}(-k) = -\hat{\hat{S}}_{CB}(k)$ and $\hat{\hat{S}}_{AB}(-k) = \hat{\hat{S}}_{AB}(k)$ by explicit calculation of the eigenvectors and appropriate normalization. Since χ_{k_0} is a symmetric function, $\hat{S}_{CB}$ and $\hat{S}_{AB}$, respectively, have the same property. Therefore, it holds that $(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k=0) = 0$. For k in a neighborhood of zero, we use that

$$\sup_{\omega\in\mathbb{R}} |\hat{S}_{CB}(k-\omega) - \hat{S}_{CB}(-\omega)| \le C|k|,$$
$$\sup_{\omega\in\mathbb{R}} |\hat{S}_{AB}(k-\omega) - \hat{S}_{AB}(-\omega)| \le C|k|$$

for $|k| \ll 1$, which can be checked using the definition of $\hat{S}$ given in (10). Using this, we obtain

$$\begin{aligned}(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k) &= \int_{\mathbb{R}} \hat{S}_{AB}(k-\omega)\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\,d\omega \\ &= \int_{\mathbb{R}} \hat{S}_{AB}(\omega)\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\,d\omega \\ &\qquad + \int_{\mathbb{R}} (\hat{S}_{AB}(k-\omega) - \hat{S}_{AB}(-\omega))\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\,d\omega\end{aligned}$$

and similarly using $f*g = g*f$,

$$\begin{aligned}(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k) &= -\int_{\mathbb{R}} \hat{S}_{CB}(\omega)\hat{p}_B(k-\omega)\hat{S}_{AB}(\omega)\hat{p}_B(\omega)\,d\omega \\ &\qquad + \int_{\mathbb{R}} (\hat{S}_{CB}(k-\omega) - \hat{S}_{CB}(k-\omega))\hat{p}_B(k-\omega)\hat{S}_{AB}(\omega)\hat{p}_B(\omega)\,d\omega\end{aligned}$$

Putting all above identities and estimates together then gives

$$|(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k)| \le K|k||\hat{p}_B|^{*2}(k)$$

for $|k| \ll 1$, which proves the statement. □

Proof of Lemma 2.5. We obtain the short-time existence of solutions of (12) using standard local existence and uniqueness theory since the system is semilinear. Therefore, the solution exists as long as (p_A, p_C, p_B, p_ψ) remain small in H^m. Then, the desired stability result follows from Lemma 2.10 by continuous induction provided that $\varepsilon > 0$ is sufficiently small, e.g. following [JZ11b]. □

3 Nonlinear stability of an invading front

In Theorem 2.6, we proved the nonlinear stability of the invading state $(1,0,0)$ of the modified Ginzburg-Landau system in polar coordinates (2) with respect to localized perturbations. Next, we study the stability of an invading front connecting this state to the origin, i.e. $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi)$ with $\xi = x - ct$ and

$$\lim_{\xi\to-\infty} (A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) = (1,0,0) \text{ and } \lim_{\xi\to 0} (A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) = (0,0,0).$$

As discussed in the introduction, such a front exists for $c \ge 2$ and γ close to zero. As highlighted above, this front connects a diffusively stable state, the invading state, to an unstable one, the origin.

We now follow the strategy outlined in the introduction. A perturbation (r_A, r_B, r_ψ) of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$ satisfies

$$\begin{aligned}\partial_t r_A &= \partial_x^2 r_A + r_A + r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3r_A A_{\text{front}}^2 - 3r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2, \\ \partial_t r_B &= \mu\partial_x^2 r_B + \gamma\partial_x^2(2r_A A_{\text{front}} + r_A^2),\end{aligned}$$

$$\partial_t r_\psi = \partial_x^2 r_\psi + 2\partial_x \left(\frac{(\partial_x A_{\text{front}} + \partial_x r_A) r_\psi}{A_{\text{front}} + r_A} \right).$$

Again, we introduce the additional variable $r_C = \partial_x r_A$ and $C_{\text{front}} = \partial_\xi A_{\text{front}}$. Then, the extended system with the unknown $r = (r_A, r_C, r_B, r_\psi)^T$ is given by

$$\begin{aligned}
\partial_t r_A &= \partial_x^2 r_A + r_A + r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3 r_A A_{\text{front}}^2 - 3 r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2, \\
\partial_t r_C &= \partial_x^2 r_C + r_C + \partial_x (r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3 r_A A_{\text{front}}^2 - 3 r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2), \\
\partial_t r_B &= \mu \partial_x^2 r_B + 2\gamma \partial_x (r_C A_{\text{front}} + r_A C_{\text{front}} + r_A r_C) \\
\partial_t r_\psi &= \partial_x^2 r_\psi + 2\partial_x \left(\frac{(\partial_x A_{\text{front}} + r_C) r_\psi}{A_{\text{front}} + r_A} \right).
\end{aligned}$$

We write the system as

$$\partial_t r = L r + \mathcal{N}_1(r) + \mathcal{N}_2(r), \tag{16}$$

where L is the same linear operator defined on $\{r = (r_A, r_C, r_B, r_\psi)^T \in \mathbb{R}^4 \,|\, \partial_x r_A = r_C\}$ as in the stability of the invading state in system (9) and $\mathcal{N}_1(r) = L_{\text{front}} - L$, where L_{front} is the linearisation about the front. Finally, $\mathcal{N}_2$ contains the remaining nonlinearities. Explicitly, $\mathcal{N}_1$ and $\mathcal{N}_2$ read as

$$\mathcal{N}_1(r) = \begin{pmatrix} r_A B_{\text{front}} + r_B (A_{\text{front}} - 1) - 3 r_A (A_{\text{front}}^2 - 1) \\ \partial_x (r_A B_{\text{front}} + r_B (A_{\text{front}} - 1) - 3 r_A (A_{\text{front}}^2 - 1)) \\ 2\gamma \partial_x (r_C (A_{\text{front}} - 1) + r_A C_{\text{front}}) \\ 2\partial_x \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} r_\psi \right) \end{pmatrix},$$

$$\mathcal{N}_2(r) = \begin{pmatrix} r_A r_B - 3 r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2 \\ \partial_x (r_A r_B - 3 r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2) \\ 2\gamma \partial_x (r_A r_C) \\ 2\partial_x \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{r_A}{A_{\text{front}} + r_A} r_\psi + \frac{r_C r_\psi}{A_{\text{front}} + r_A} \right) \end{pmatrix}.$$

Similar to [ES02], it turns out that $\mathcal{N}_1$, which is linear in r, can be treated as an additional nonlinearity in appropriately weighted spaces.

3.1 Weighted operator and linear estimates

As discussed in the introduction, we introduce a weighted variable $w_j(\xi, t) = e^{\beta\xi} r_j(\xi + ct, t)$ for $j \in \{A, C, B\}$ and $w_\psi(\xi, t) = e^{\beta_\psi \xi} r_\psi(\xi + ct, t)$, where $\xi = x - ct$ and weights $\beta, \beta_\psi > 0$ to be determined later. In the co-moving frame, these variables satisfy

$$\begin{aligned}
\partial_t w_A &= \partial_\xi^2 w_A + (c - 2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1) w_A + A_{\text{front}} w_B + B_{\text{front}} w_A \\
&\quad + w_B r_A - 3 A_{\text{front}}^2 w_A - 3 A_{\text{front}} w_A r_A - r_A^2 w_A - e^{(\beta - \beta_\psi)\xi} A_{\text{front}} w_\psi r_\psi - w_A r_\psi^2, && \text{(17a)} \\
\partial_t w_C &= \partial_\xi^2 w_C + (c - 2\beta)\partial_\xi w_C + (\beta^2 - c\beta + 1) w_C + (\partial_\xi - \beta)(A_{\text{front}} w_B + B_{\text{front}} w_A) \\
&\quad + (\partial_\xi - \beta)(w_B r_A - 3 A_{\text{front}}^2 w_A - 3 A_{\text{front}} w_A r_A - r_A^2 w_A - w_A r_\psi^2) - e^{(\beta - \beta_\psi)\xi} (\partial_\xi - \beta_\psi)(A_{\text{front}} w_\psi r_\psi), && \text{(17b)} \\
\partial_t w_B &= \mu \partial_\xi^2 w_B + (c - \mu 2\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta) w_B + 2\gamma(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A + w_C r_A), && \text{(17c)} \\
\partial_t w_\psi &= \partial_\xi^2 w_\psi + (c - 2\beta_\psi)\partial_\xi w_\psi + (\beta_\psi^2 - c\beta_\psi) w_\psi + 2(\partial_\xi - \beta_\psi) \left(\frac{\partial_\xi A_{\text{front}} + r_C}{A_{\text{front}} + r_A} w_\psi \right). && \text{(17d)}
\end{aligned}$$

We abbreviate the above system (17) by

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, r), \tag{18}$$

where L_β contains the linear and $\mathcal{N}_\beta$ the nonlinear terms. Note that, since $\partial_x r_A = r_C$ we obtain $(\partial_\xi - \beta) w_A = w_C$ and thus, we restrict L_β to the set $\{w = (w_A, w_C, w_B, w_\psi)^T \,|\, (\partial_\xi - \beta) w_A = w_C\}$ in what follows. To prove the stability result, we use the following property on the linear operator L_β, which we prove in Section 4.

Lemma 3.1. *Let $\beta > 0$, $\mu > 0$ and $c > c_{min}(\mu)$, see (4), such that*

$$\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) < 0. \tag{19}$$

Then, there exists a $\beta_\psi^0 > 0$, depending on c, such that for every $\beta_\psi \in (0, \beta_\psi^0)$ the operator $L_\beta : \mathcal{D}(L_\beta) \to L^2$ is a densely defined, closed, sectorial operator with domain $\mathcal{D}(L_\beta) \subset L^2$. Furthermore, there exists a $\gamma_0 > 0$ and a $\kappa > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds

1. $\mathrm{Re}(\sigma(L_\beta)) \leq -\kappa$ *and*
2. *there exists a $\kappa \geq \tilde{\kappa} > 0$ such that the estimates*

$$\left\| e^{tL_\beta} w \right\|_{L^2} \lesssim e^{-\tilde{\kappa} t} \left\| w \right\|_{L^2}, \tag{20}$$

$$\left\| e^{tL_\beta} \partial_\xi w \right\|_{L^2} \lesssim e^{-\tilde{\kappa} t} (1 + t^{-1/2}) \left\| w \right\|_{L^2}, \tag{21}$$

hold for any $t > 0$ and $w \in L^2$.

Remark 3.2. In particular, Lemma 3.1 implies that there is no eigenvalue at $\lambda = 0$, which would be expected since the original system (16) is translational invariant. Therefore, we formally have that

$$L_\beta \begin{pmatrix} \partial_\xi A_{\text{front}} e^{\beta\xi} \\ \partial_\xi C_{\text{front}} e^{\beta\xi} \\ \partial_\xi B_{\text{front}} e^{\beta\xi} \\ 0 \end{pmatrix} = 0.$$

However, we cannot expect the derivative of $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ to decay fast enough such that the potential eigenfunction is bounded. This follows by insertion of $(A, B)(x, t) = (\tilde{A}, \tilde{B})(x - ct)$ into (1) and considering the eigenvalues of the linearisation about the origin in the resulting first-order ODE-system, see (34). In this system, the origin is a stable, hyperbolic fixed-point and the largest eigenvalue is given by

$$\lambda_{\max} := \max\left(-\frac{c}{\mu}, \frac{1}{2}\left(-c + \sqrt{c^2 - 4}\right)\right).$$

In particular, we have $|\lambda_{\max}| < \beta$, if β satisfies (19). Hence, we expect that generically $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ converges to the origin slower than $e^{-\beta\xi}$ for $\xi \to \infty$. Thus, the formal eigenfunction is unbounded and in particular not in L^2. □

Remark 3.3. We also highlight that we can choose β and β_ψ in Lemma 4.3 such that $e^{(\beta - \beta_\psi)\xi} A_{\text{front}}$ and $e^{(\beta - \beta_\psi)\xi} (\partial_\xi - \beta_\psi) A_{\text{front}}$ in (17) are bounded. To show this we note that it is sufficient to proof boundedness for $\xi \to \infty$. Using Proposition A.1 we therefore need to choose the weights β and β_ψ such that

$$\beta - \beta_\psi - \frac{1}{2}(c - \sqrt{c^2 - 4}) \leq 0. \tag{22}$$

Utilizing that (19) is equivalent to

$$\frac{1}{2}(c - \sqrt{c^2 - 4}) < \beta < \min\left(\frac{c}{\mu}, \frac{1}{2}(c + \sqrt{c^2 - 4})\right),$$

condition (22) can be satisfied by choosing β close to $(c - \sqrt{c^2 - 4})/2$. □

3.2 Nonlinear stability

We stress that Lemma 3.1 only makes statements on estimates in L^2 instead of H^m. However, this is enough to obtain the necessary results in H^m. For that we prove a nonlinear damping estimate similar to [JZ11a, JZ11b], which controls the H^m-norm of any solution w of (18) by its L^2-norm and the H^m-norm of the initial data $w|_{t=0}$ as long as all variables stay sufficiently small in H^m.

Lemma 3.4. *Let $m \in \mathbb{N}$ and assume that $w|_{t=0} \in H^m$ and that there exists some $T > 0$ such that the H^m-norm of r and w remains sufficiently bounded for all $0 \le t \le T$. In particular, assume that there exists a constant $K_{r,T}$ such that $\sup_{0\le t\le T} \|r(t)\|_{H^m} \le K_{r,T}$. Then there exists constants $\theta_m > 0$ and $K < \infty$ depending on $K_{r,T}$ such that*

$$\|w(t)\|_{H^m}^2 \le e^{-\theta_m t} \|w|_{t=0}\|_{H^m}^2 + K \int_0^t e^{-\theta_m (t-s)} \|w(s)\|_{L^2}^2 \, ds \tag{23}$$

for all $0 \le t \le T$.

Proof. We take the L^2 scalar product of (18) with the test function $\vartheta = \sum_{j=0}^m (-1)^j \partial_\xi^{2j} w$, i.e.

$$\langle \vartheta, \partial_t w \rangle_{L^2} = \langle \vartheta, L_\beta w + \mathcal{N}_\beta(w,r) \rangle_{L^2} .$$

By choice of the test function, it holds that $2 \langle \vartheta, \partial_t w \rangle_{L^2} = \partial_t \|w\|_{H^m}^2$ and $\left\langle \vartheta, \partial_\xi^2 w \right\rangle_{L^2} = \|w\|_{H^m}^2 + \left\| \partial_\xi^{m+1} w \right\|_{L^2}^2$. Furthermore, for $f \in C_b^{m+1}$, $n \in \{0,1,2\}$, $0 \le j \le m$ and $i_1, i_2, i_3 \in \{A, C, B, \psi\}$ we estimate with Young's inequality

$$\begin{aligned}\left\langle (-1)^j \partial_\xi^{2j} w_{i_1}, f w_{i_2} r_{i_3}^n \right\rangle_{L^2} &= \left\langle \partial_\xi^j w_{i_1}, \partial_\xi^j (f w_{i_2} r_{i_3}^n) \right\rangle_{L^2} \lesssim \left\| \partial_\xi^j w_{i_1} \right\|_{L^2}^2 + \|f\|_{C_b^m}^2 \|w_{i_2}\|_{H^m}^2 \|r_{i_3}\|_{H^m}^{2n} \\ &\lesssim \left\| \partial_\xi^j w_{i_1} \right\|_{L^2}^2 + \|f\|_{C_b^m}^2 K_{r,T}^{2n} \|w_{i_2}\|_{H^m}^2 ,\end{aligned}$$

where we used the fact that H^m is a Banach algebra for $m > 1/2$ and that w and r are assumed to be small in H^m. Similarly, for any $\varepsilon > 0$ holds that

$$\left\langle (-1)^j \partial_\xi^{2j} w_{i_1}, \partial_\xi (f w_{i_2} r_{i_3}^n) \right\rangle_{L^2} = -\left\langle \partial_\xi^{j+1} w_{i_1}, \partial_\xi^j (f w_{i_2} r_{i_3}^n) \right\rangle_{L^2} \lesssim \varepsilon \left\| \partial_\xi^{j+1} w_{i_1} \right\|_{L^2}^2 + \frac{1}{\varepsilon} \|f\|_{C_b^m}^2 K_{r,T}^{2n} \|w_{i_2}\|_{H^m}^2 ,$$

where we used Young's inequality. Using Remark 3.3 and the fact r_A decays faster than A_{front} for $\xi \to \infty$ since w_A is small in H^m and thus in L^∞ we can treat all but one terms in (17) using the above estimate. To treat the remaining term, let $\chi_0 \in [0,1]$ be a smooth cut-off function with $\chi_0(\xi) = 0$ for $\xi \le -1$ and $\chi_0(\xi) = 1$ for $\xi \ge 1$. Then, we write

$$\left\langle (-1)^j \partial_\xi^{2j} w_\psi, \partial_\xi \left(\frac{r_C}{A_{\text{front}} + r_A} w_\psi \right) \right\rangle = -\left\langle \partial_\xi^{j+1} w_\psi, \partial_\xi^j \left(\chi_0 \frac{w_C w_\psi}{A_{\text{front}} e^{\beta\xi} + w_A} + (1-\chi_0) \frac{r_C w_\psi}{A_{\text{front}} + r_A} \right) \right\rangle$$

Since r_C is small in H^m and thus small in L^∞ for all $0 \le t \le T$, we obtain that $(A_{\text{front}} + r_A)|_{\xi \in (-\infty,1]}$ is bounded away from zero. Furthermore, using Proposition A.1 and the assumption that w_A is small in H^m we find that $(A_{\text{front}} e^{\beta\xi} + w_A)|_{\xi \in [-1,\infty)}$ is bounded away from zero. Thus, we estimate

$$\left\langle (-1)^j \partial_\xi^{2j} w_\psi, \partial_\xi \left(\frac{r_C}{A_{\text{front}} + r_A} w_\psi \right) \right\rangle \lesssim \frac{1}{\varepsilon} (\|w_C\|_{H^m} + \|r_c\|_{H^m})^2 \|w_\psi\|_{H^m}^2 + \varepsilon \left\| \partial_\xi^{j+1} w_\psi \right\|_{L^2}^2$$

for some small $\varepsilon > 0$, using Young's inequality.

Hence, combining the above estimates and choosing $\varepsilon > 0$ arbitrarily small, there exists a $\tilde{\theta}_m > 0$ and a constant $\tilde{K} > 0$, both depending on $K_{r,T}$, such that

$$\partial_t \|w(t)\|_{H^m}^2 \le -\tilde{\theta}_m \left\| \partial_\xi^{m+1} w(t) \right\|_{L^2}^2 + \tilde{K} \|w(t)\|_{H^m}^2$$

for all $0 \le t \le T$. Then, using Sobolev interpolation for every $\tilde{\varepsilon} > 0$ exists a $K_{\tilde{\varepsilon}} > 0$ such that

$$\|g\|_{H^m} \le \tilde{\varepsilon} \left\|\partial_\xi^{m+1} g\right\|_{L^2}^2 + K_{\tilde{\varepsilon}} \|g\|_{L^2}^2 . \tag{24}$$

Inserting this into the above equation and choosing $\tilde{\varepsilon}$ small enough we find that there exists a $\theta_m > 0$ and a constant $K > 0$ such that

$$\partial_t \|w(t)\|_{H^m}^2 \le -\theta_m \|w(t)\|_{H^m}^2 + K \|w(t)\|_{L^2}^2$$

for all $0 \le t \le T$. Again, both constants depend on $K_{r,T}$. Then, the statement follows by applying Gronwall's inequality. □

Remark 3.5. Using the nonlinear damping estimate Lemma 3.4 and the semigroup estimates Corollary 2.4 and Lemma 3.1, we obtain local existence and uniqueness of the coupled (r, w)-system (16), (18) in H^m using standard fixed point arguments. □

Using Lemma 3.1 we can now prove the nonlinear stability. Therefore, we introduce

$$\mu_j(t) = \sup_{0 \le s \le t} e^{\bar{\kappa} s} \|w_j(s)\|_{H^m}$$

for $j \in \{A, C, B, \psi\}$ and $2\bar{\kappa} \in (0, \min(\theta_m, 2\tilde{\kappa}))$ with $\tilde{\kappa} > 0$ from Lemma 3.1 and θ_m from Lemma 3.4. Then, we consider the weighted equation (18) and use $r = Sp$, with S from (10), to obtain

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, Sp),$$

where p satisfies $\partial_t p = \Lambda p + S^{-1}\mathcal{N}_1(Sp) + S^{-1}\mathcal{N}_2(Sp)$. Again any solution of this satisfies the Duhamel formula

$$w(t) = e^{tL_\beta} w|_{t=0} + \int_0^t e^{(t-s)L_\beta} \mathcal{N}_\beta(w(s), Sp(s))\, ds$$

for all $t \ge 0$. Using (20), (21), Lemma 3.4, the estimate

$$e^{\bar{\kappa} t} \int_0^t e^{-\tilde{\kappa}(t-s)} \max(1, (t-s)^{-1/2}) e^{-\bar{\kappa} s}\, ds < \infty$$

and the fact that $\mathcal{N}_\beta(w, Sp)$ is linear with respect to w we obtain the following result.

Proposition 3.6. *Fix $m \in \mathbb{N}$. Then, let $\beta > 0$, $\beta_\psi > 0$, $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$ such that Lemma 3.1 applies and (22) holds. Furthermore, assume that there exists a $T > 0$ such that $w(t)$ and $r(t) = Sp(t)$ are small in H^m for all $0 \le t \le T$. Then, $E_w(t) := \mu_A(t) + \mu_C(t) + \mu_B(t) + \mu_\psi(t)$ satisfies the estimate*

$$E_w(t) \le K_w(\|w|_{t=0}\|_{H^m} + E_w(t)E(t) + E_w(t)^2), \tag{25}$$

with $K_w < \infty$, $E(t) := \eta_A(t) + \eta_C(t) + \eta_B(t) + \widetilde{\eta_B}(t) + \eta_\psi(t)$, with η_j, $\widetilde{\eta_B}$ defined in (14), and θ_m from Lemma 3.4.

Proof. Analogue to Lemma 2.10, using (20), (21) and $\bar{\kappa} < \tilde{\kappa}$ we obtain the estimate

$$\|w(t)\|_{L^2} \lesssim e^{-\bar{\kappa} t} \left(\|w|_{t=0}\|_{L^2} + E_w(t)E(t) + E_w(t)^2\right),$$

since r is small in H^m. The only additional challenge originates from the $w_\psi(\partial_\xi A_{\text{front}} + r_C)/(A_{\text{front}} + r_A)$-term in the w_ψ equation. However, this can be handled using a similar splitting argument as in Lemma 3.4, which leads to the quadratic contribution $E_w(t)^2$. Note that the main challenge of Lemma 2.10 – the polynomially decaying variable – does not appear here since we have exponential decay of the semigroup

generated by L_β, see Lemma 3.1. Then, using the damping estimate Lemma 3.4 yields the statement, since we have that

$$\int_0^t e^{-\theta_m(t-s)} e^{-2\bar{\kappa}s}\, ds = \mathcal{O}(e^{-2\bar{\kappa}t}),$$

since $2\bar{\kappa} < \theta_m$ by assumption. □

We now treat $\mathcal{N}_1$. A closer inspection reveals that all terms in $\mathcal{N}_1$ are of the form $f(\xi)r_j(x,t)$, $j \in \{A, C, B, \psi\}$, where f decays exponentially fast for $\xi \to -\infty$. Ideally, if $f(\xi)e^{-\beta_j\xi} = \mathcal{O}(1)$ for $\xi \to -\infty$ we would write $f(\xi)r_j(x,t) = f(\xi)w_j(\xi,t)e^{-\beta_j\xi}$ and use that w decays exponentially, at least on the linear level. However, we can only verify this for the ψ-part of $\mathcal{N}_1$, for which f is given by

$$f_\psi = \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}.$$

Since we can choose β_ψ close to zero, we obtain that $f_\psi(\xi)e^{-\beta_\psi\xi}$ is bounded for $\xi \to -\infty$. For the remaining part, we cannot expect that a similar statement is true since its in general false for the KPP-equation, which corresponds to the case $\gamma = 0$, $r_B = r_\psi = 0$ in our setting. In the case of the KPP-equation the asymptotic behavior of the fronts for $\xi \to -\infty$ can be explicitly calculated, see [Sat76] and we obtain that $(1 - A_{\text{front}})e^{-\beta\xi}$ is unbounded for any appropriate choice of β. However, it turns out that we do not need that $f(\xi)e^{-\beta\xi}$ is bounded. Instead we use the following result.

Lemma 3.7. *Let $m \in \mathbb{N}_0$, $\tilde{c} > 0$ be small and $f \in C_b^m$ with $f(\xi) = \mathcal{O}(e^{-\tilde{\theta}\xi})$ for $\xi \to -\infty$ for some $\tilde{\theta} > 0$. Then, for all $\delta > 0$ exists a constant $K_{\tilde{c},\delta} < \infty$ such that*

$$\|f(x-ct)r_j(x,t)\|_{H^m(x)} \le \delta e^{-\tilde{\theta}\tilde{c}t}\, \|r_j(x,t)\|_{H^m(x)} + K_{\tilde{c},\delta} e^{\beta_j\tilde{c}t}\, \|w_j(\xi,t)\|_{H^m(\xi)}$$

for any $j \in \{A, C, B, \psi\}$.

Proof. First note that since $\partial_\xi = \partial_x$ we have $\|g(x-ct)\|_{C_b^m(x)} = \|g(\xi)\|_{C_b^m(\xi)}$ if $g \in C_b^m$. Next, let χ_{ξ_0} be a smooth cutoff function with $\chi_{\xi_0} \in [0,1]$, $\chi_{\xi_0}(\xi) = 0$ for $\xi > \xi_0 + 1$ and $\chi_{\xi_0}(\xi) = 1$ for $\xi < \xi_0 - 1$ for a fixed $\xi_0 \in \mathbb{R}$. Then, we write for $j \in \{A, C, B, \psi\}$

$$\begin{aligned} f(\xi)r_j(x,t) &= \chi_{\xi_0-\tilde{c}t}(\xi)f(\xi)r_j(x,t) + (1-\chi_{\xi_0-\tilde{c}t}(\xi))f(\xi)e^{-\beta_j\xi}w_j(\xi,t) \\ &= \chi_{\xi_0-\tilde{c}t}(\xi)f(\xi)r_j(x,t) + (1-\chi_{\xi_0-\tilde{c}t}(\xi))f(\xi)e^{-\beta_j(\xi+\tilde{c}t)}e^{\beta_j\tilde{c}t}w_j(\xi,t). \end{aligned}$$

The lemma is proven if the estimates

$$\begin{aligned} \|\chi_{\xi_0-\tilde{c}t}(\xi)f(\xi)\|_{C_b^m(\xi)} &\le \delta e^{-\tilde{\theta}\tilde{c}t}, \\ \sup_{t\ge 0}\left\|(1-\chi_{\xi_0-\tilde{c}t}(\xi))f(\xi)e^{-\beta_j(\xi+\tilde{c}t)}\right\|_{C_b^m(\xi)} &\le K_{\tilde{c},\delta} \end{aligned}$$

hold. To prove these inequalities we introduce

$$\begin{aligned} \mathcal{U}_{\xi_0}(t) &= \operatorname{supp}(\chi_{\xi_0-\tilde{c}t}) = \{\xi \in \mathbb{R} : \xi \le \xi_0 - \tilde{c}t + 1\}, \\ \tilde{\mathcal{U}}_{\xi_0}(t) &= \operatorname{supp}(1-\chi_{\xi_0-\tilde{c}t}) = \{\xi \in \mathbb{R} : \xi \ge \xi_0 - \tilde{c}t - 1\}. \end{aligned}$$

Then, the first estimate follows from

$$\|\chi_{\xi_0-\tilde{c}t}(\xi)f(\xi)\|_{C_b^m} \le \|\chi_{\xi_0-\tilde{c}t}\|_{C_b^m}\, \|f(\xi)\|_{C_b^m(\mathcal{U}_{\xi_0-\tilde{c}t}(t))} \le \delta e^{-\tilde{\theta}(\xi_0+\tilde{c}t+1)},$$

which holds for any small $\delta > 0$ by choosing $\xi_0 = \xi_0(\delta) < 0$ small enough. Furthermore, we obtain the second estimate by

$$\sup_{t\ge 0}\left\|(1-\chi_{\xi_0-\tilde{c}t}(\xi))f(\xi)e^{-\beta_j(\xi+\tilde{c}t)}\right\|_{C_b^m} \le \|f\|_{C_b^m} \sup_{t\ge 0}\left\|e^{-\beta_j(\xi+\tilde{c}t)}\right\|_{C_b^m(\tilde{\mathcal{U}}(\chi_{\xi_0-\tilde{c}t}(t)))}$$

$$\leq K(1+\beta_j^m)\sup_{t\geq 0} e^{-\beta_j(\xi_0-\tilde{c}t-1+\tilde{c}t)} = K(1+\beta_j^m)e^{-\beta_j(\xi_0-1)} < \infty.$$

Note that for $\xi_0 \to -\infty$, this bound tends to $+\infty$. □

We now comment on why this is enough to treat $\mathcal{N}_1$ as a nonlinear term in the stability analysis. The above Lemma 3.7 states that we can estimate every term in $\mathcal{N}_1$ from above by the H^m-norm of r with an arbitrarily small, exponentially decaying prefactor and the H^m-norm of w with a prefactor which grows exponentially fast. However, using the linear stability estimates in Lemma 3.1 and the nonlinear damping estimate Lemma 3.4 we can expect that the H^m-norm of w shows exponential decay. Thus, by choosing $\tilde{c} > 0$ small enough it is, at least heuristically, possible to obtain an exponentially decaying estimate, which allows to treat $\mathcal{N}_1$ as a nonlinearity, see the proof of the subsequent Lemma 3.8 for details.

We now have all necessary results to prove nonlinear stability of the fronts in the system

$$\partial_t p = \Lambda p + S^{-1}\mathcal{N}_1(Sp) + S^{-1}\mathcal{N}_2(Sp), \tag{26a}$$

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, Sp), \tag{26b}$$

with Λ, S, S^{-1} from Lemma 2.1 and (10). Then, the main result reads as follows.

Lemma 3.8. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$, see (4). Then, choose $\beta > 0$ and $\beta_\psi > 0$ such that*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta + c) < 0,\\
&\beta_\psi \in (0, \beta_\psi^0),\\
&\beta \leq \frac{1}{2}(c - \sqrt{c^2-4}) + \beta_\psi
\end{aligned}$$

with β_ψ^0 from Lemma 3.1. Then, there exists $\varepsilon, \varepsilon_w > 0$ and a $\bar{\kappa} > 0$ such that if

$$\begin{aligned}
\|(p_A, p_C, p_B, p_\psi)|_{t=0}\|_{H^m\times H^m\times(L^1\cap H^m)\times(L^1\cap H^m)} &< \varepsilon,\\
\|(w_A, w_C, w_B, w_\psi)|_{t=0}\|_{H^m\times H^m\times H^m\times H^m} &< \varepsilon_w,
\end{aligned}$$

there exists a constant $K < \infty$ such that

$$E(t) + E_w(t) \leq K$$

for all $t \geq 0$. In particular, this yields

$$\|p_A(t)\|_{H^m}, \|p_C(t)\|_{H^m} \lesssim (1+t)^{-1/2},\ \|p_B(t)\|_{H^m}, \|p_\psi(t)\|_{H^m} \lesssim (1+t)^{-1/4},\ \text{and}\ \|w(t)\|_{H^m} \lesssim e^{-\bar{\kappa}t}$$

for all $t \geq 0$.

Proof. Since the system (26) is semilinear and using the semigroup estimates (20), (21) for L_β provided in Lemma 3.1, the semigroup estimates for Λ from Corollary 2.4 and the nonlinear damping estimate Lemma 3.4, we obtain local existence and uniqueness in H^m by standard arguments. In particular, E and E_w are continuous with respect to time as long as they remain small. Hence, for sufficiently small initial data, there exists a time $T > 0$ such that

$$\sup_{0\leq t\leq T} E(t) \leq \frac{1}{2K_w}. \tag{27}$$

In particular, since $\hat{S}$ is bounded in Fourier space, this yields that $Sp(t)$ is uniformly bounded in H^m for all $0 \leq t \leq T$. Therefore, we can apply Proposition 3.6 and obtain

$$E_w(t) \leq 2K_w(\|w|_{t=0}\|_{H^m} + E_w(t)^2).$$

for all $0 \le t \le T$, which yields that

$$E_w(t) \le \tilde{K}_w \varepsilon_w \tag{28}$$

for all $0 \le t \le T$. Then, we consider the Duhamel formula for (26a)

$$p(t) = e^{t\Lambda} p|_{t=0} + \int_0^t e^{(t-s)\Lambda} S^{-1} \mathcal{N}_1(Sp(s))\, ds + \int_0^t e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s))\, ds.$$

We estimate the two integral terms separately. For this, we note that $(A_{\text{front}} - 1), C_{\text{front}}, B_{\text{front}}$ satisfy the assumptions of Lemma 3.7 for some $\theta > 0$. Hence, for $\tilde{c} > 0$ small we estimate the integral term for $\mathcal{N}_1$ using Lemmas 2.1 and 3.7. Similar to Lemma 2.10 we focus on the polynomially decaying part of the estimates for p_B, $S_{CB} p_B$ and p_ψ since all other components have exponential decay on the linear level, which simplifies the estimates. Therefore, let P_j, $j = A, C, B, \psi$ be the projection onto the j-th component as in Corollary 2.4.

We start by estimating p_B. Using Lemma 2.1, Corollary 2.4, Lemma 3.7 and $r = Sp$ we find

$$\begin{aligned}
&\int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1} \mathcal{N}_1(Sp(s)) \right\|_{H^m} ds \\
&\quad \lesssim \int_0^t e^{-\theta(t-s)} \max(1, (t-s)^{-1/2}) \left(\delta e^{-\tilde{\theta}\tilde{c}s} \|p(s)\|_{H^m} + K_{\tilde{c},\delta} e^{\beta\tilde{c}s} \|w(s)\|_{H^m} \right) ds \\
&\qquad + \max(1, 2|\gamma|) \int_0^t (t-s)^{-1/2} \left(\delta e^{-\tilde{\theta}\tilde{c}s} \|p(s)\|_{H^m} + K_{\tilde{c},\delta} e^{\beta\tilde{c}s} \|w(s)\|_{H^m} \right) ds \\
&\quad \lesssim \max(1, 2|\gamma|)(1+t)^{-1/2} (\delta E(t) + K_{\tilde{c},\delta} E_w(t))
\end{aligned}$$

Following the proof of Lemma 2.5 and using that $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ are uniformly bounded in C_b^m, we obtain for $\mathcal{N}_2$ that

$$\int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/4} E(t)^2.$$

Similarly, we estimate $S_{CB} p(t)$ in H^m. For the $\mathcal{N}_1$ estimate we use that S_{CB} is a bounded map from H^m to H^m and thus, we obtain the same estimate as above for $p_B(t)$. For $\mathcal{N}_2$ we use as in Lemma 2.5 that $|\hat{S}_{CB}(k)| = \mathcal{O}(|k|)$ for $|k| \to 0$, which gives an additional $(1+t-s)^{-1/2}$ in the semigroup estimates.

Finally, we estimate $p_\psi(t)$ in H^m. For the $\mathcal{N}_1$-term we use that $\partial_\xi A_{\text{front}} / A_{\text{front}}$ satisfies the assumptions of Lemma 3.7 and thus proceeding as above we find

$$\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_1(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/2} (\delta E(t) + K_{\tilde{c},\delta} E_w(t)).$$

To estimate the $\mathcal{N}_2$-contribution, let χ_0 be a cut-off function with $\chi_0 \in [0,1]$, $\chi_0(\xi) = 0$ for $\xi \le -1$ and $\chi_0(\xi) = 1$ for $\xi \ge 1$. Then, proceeding with a similar splitting as in the proof of Lemma 3.4 we find

$$\begin{aligned}
&\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \\
&\quad = \int_0^t \left\| e^{(t-s)\partial_x^2} 2\partial_x \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{r_\psi(s) r_A(s)}{A_{\text{front}} + r_A(s)} + \frac{r_C(s) r_\psi(s)}{A_{\text{front}} + r_A(s)} \right) \right\|_{H^m} ds \\
&\quad \lesssim \int_0^t (t-s)^{-1/2} \left\| \chi_0 \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{w_A(s) r_\psi(s)}{A_{\text{front}} e^{\beta\xi} + w_A(s)} + \frac{w_C(s) r_\psi(s)}{A_{\text{front}} e^{\beta\xi} + w_A(s)} \right) \right\|_{H^m} ds \\
&\qquad + \int_0^t \left\| e^{(t-s)\partial_x^2} \partial_x \left((1-\chi_0) \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{r_A r_\psi}{A_{\text{front}} + r_A} \right) \right\|_{H^m} ds
\end{aligned}$$

$$+ \int_0^t \left\| e^{(t-s)\partial_x^2} \partial_x \left((1-\chi_0) \frac{r_C r_\psi}{A_{\text{front}} + r_A} \right) \right\|_{H^m} ds$$
$$=: I_1 + I_2 + I_3.$$

To estimate I_1, we use that since $w_A(s)$ is small in H^m for $0 \le s \le T$, $A_{\text{front}} e^{\beta\xi} + w_A(s)$ is bounded away from zero on $[-1, \infty)$ for all $0 \le s \le T$. This yields

$$I_1 \lesssim \int_0^t (t-s)^{-1/2} (\|w_A(s)\|_{H^m} + \|w_C(s)\|_{H^m}) \|r_\psi(s)\|_{H^m} \, ds \lesssim (1+t)^{-1/2} E_w(t) E(t).$$

For I_2 we use that $\partial_\xi A_{\text{front}} / A_{\text{front}}$ satisfies the assumption of Lemma 3.7 and that $A_{\text{front}} + r_A$ is bounded away from zero on $(-\infty, 1]$, which leads to

$$I_2 \lesssim \int_0^t (t-s)^{-1/2} \|r_A(s)\|_{H^m} \left\| \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} r_\psi(s) \right\|_{H^m} ds \lesssim (1+t)^{-1/2} \delta E(t)^2 + K_{\tilde{c},\delta} E_w(t) E(t).$$

Finally, we obtain for I_3 that

$$I_3 \lesssim (1+t)^{-1/4} E(t)^2$$

similar to the proof of Lemma 2.5 by using the improved decay rate for $S_{CB} p_B$. Combining the estimates for I_1, I_2 and I_3 then yields

$$\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/4} (E(t) E_w(t) + E(t)^2).$$

Putting the above estimates together and using the linear stability estimates in Corollary 2.4 we obtain

$$\begin{aligned} E(t) &\le K(\varepsilon + \delta E(t) + K_{\tilde{c},\delta} E_w(t) + E(t) E_w(t) + E(t)^2) \\ &\le K(\varepsilon + (\delta + \tilde{K}_w \varepsilon_w) E(t) + K_{\tilde{c},\delta} \tilde{K}_w \varepsilon_w + E(t)^2) \end{aligned}$$

for all $0 \le t \le T$, where we used (28). Since $\delta > 0$ and $\varepsilon_w > 0$ are arbitrary, this yields

$$\begin{aligned} E(t) &\le 2K(\varepsilon + K_{\tilde{c},\delta} \tilde{K}_w \varepsilon_w + E(t)^2) \\ &\le \tilde{K}(\varepsilon + \varepsilon_w + E(t)^2) \end{aligned}$$

for all $0 \le t \le T$. In particular, by choosing $\varepsilon > 0$ and $\varepsilon_w > 0$ small enough and using continuous induction, we can extend the argument to $T = \infty$, which proves the statement. □

Using that S, S^{-1} are bounded transformations in H^m, see Lemma 2.1, we obtain the stability of fronts in the system (1) with respect to real perturbations similarly to the proof of Theorem 2.6.

Theorem 3.9. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$. Then, choose $\beta > 0$ and $\beta_\psi > 0$ such that*

$$\begin{aligned} &\max(\beta^2 - c\beta + 1, \mu\beta + c) < 0, \\ &\beta_\psi \in (0, \beta_\psi^0), \\ &\beta \le \frac{1}{2}(c - \sqrt{c^2 - 4}) + \beta_\psi, \end{aligned}$$

with β_ψ^0 from Lemma 3.1. Then, there exists a $\varepsilon > 0$, $\varepsilon_w > 0$ and a $\bar{\kappa} > 0$ such that if

$$\begin{aligned} &\|(r_A, r_B, r_\psi)|_{t=0}\|_{H_2^{m+1} \times H_2^m \times (L^1 \cap H^m)} < \varepsilon \\ &\left\|(r_A, r_B)|_{t=0} e^{\beta x}\right\|_{H^{m+1} \times H^m} + \left\|r_\psi|_{t=0} e^{\beta_\psi x}\right\|_{H^m} < \varepsilon_w, \end{aligned}$$

and $\gamma \in (-\gamma_0, \gamma_0)$ *it holds that*

$$\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4},$$
$$\left\|(r_A, r_B)(t)e^{\beta(x-ct)}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi(t)e^{\beta_\psi(x-ct)}\right\|_{H^m} \lesssim e^{-\bar{\kappa}t}$$

for all $t \geq 0$.

4 Spectral and linear stability of the weighted operator

We now provide a proof of Lemma 3.1. The approach is based on perturbation arguments and estimates on the spectrum in the case $\gamma = 0$.

4.1 Spectral estimates for L_β for γ close to zero

Recall that the weighted operator L_β is given by

$$\begin{aligned} L_\beta w = &\begin{pmatrix} \partial_\xi^2 w_A + (c-2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1)w_A + A_{\text{front}} w_B - 3A_{\text{front}}^2 w_A \\ \partial_\xi^2 w_C + (c-2\beta)\partial_\xi w_C + (\beta^2 - c\beta + 1)w_C + (\partial_\xi - \beta)(A_{\text{front}} w_B - 3A_{\text{front}}^2 w_A) \\ \mu\partial_\xi^2 w_B + (c-2\mu\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta)w_B \\ \partial_\xi^2 w_\psi + (c-2\beta)\partial_\xi w_\psi + (\beta_\psi^2 - c\beta_\psi)w_\psi + 2(\partial_\xi - \beta_\psi)(\partial_\xi A_{\text{front}}/A_{\text{front}})w_\psi \end{pmatrix} \\ &+ \begin{pmatrix} B_{\text{front}} w_A \\ (\partial_\xi - \beta)(B_{\text{front}} w_A) \\ 2\gamma(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A) \\ 0 \end{pmatrix} \\ =: &\begin{pmatrix} \widetilde{L_\beta} & 0 \\ 0 & L_\psi \end{pmatrix} w + \begin{pmatrix} P(\gamma) & 0 \\ 0 & 0 \end{pmatrix} w \end{aligned}$$

for $w = (w_A, w_C, w_B, w_\psi) \in \mathcal{D}(L_\beta)$ and $\widetilde{L_\beta}, P(\gamma)$ operate on (w_A, w_C, w_B). Using the diagonal structure of L_β, we can analyze the spectra of $\widetilde{L_\beta} + P(\gamma)$ and L_ψ separately.

For $\widetilde{L_\beta} + P(\gamma)$ we use perturbation arguments. Since the $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ are uniformly bounded and B_{front} vanishes uniformly in C^1 for $\gamma \to 0$, there exists constants $a(\gamma), b(\gamma)$, which vanish for $\gamma \to 0$ and satisfy

$$\|P(\gamma)\tilde{w}\|_{L^2} \leq a(\gamma)\|\tilde{w}\|_{L^2} + b(\gamma)\left\|\widetilde{L_\beta}\tilde{w}\right\|_{L^2} \tag{29}$$

for all $\tilde{w} \in \mathcal{D}(\widetilde{L_\beta})$. Thus, by [Kat66, Theorem IV.3.17] the spectrum of $\widetilde{L_\beta} + P(\gamma)$ is close to the spectrum of $\widetilde{L_\beta}$ if γ is close to zero. Additionally, we have the following result on the spectrum of $\widetilde{L_\beta}$.

Lemma 4.1. *Let* $\beta > 0$, $\mu > 0$ *and* $c > c_{min}(\mu)$, *see* (4), *and suppose that there exists a* $\kappa > 0$ *such that*

$$\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) \leq -\kappa < 0.$$

Then, $\text{Re}(\sigma(\widetilde{L_\beta})) \leq -\kappa$.

Proof. The proof is done in two steps: first, we analyze the essential spectrum and afterwards the point spectrum.

To control the essential spectrum, we note that it is sufficient to consider the asymptotic operators

$$\widetilde{L_{\beta,\pm}}(\partial_\xi) := \lim_{\xi\to\pm\infty} \widetilde{L_\beta}(\xi, \partial_\xi).$$

Since A_{front} convergences exponentially fast to its rest states we can bound the essential spectrum of $\widetilde{L_\beta}$ by the essential spectrum of the asymptotic operators, see [San02]. We first consider $\widetilde{L_{\beta,+}}$, which is given by

$$\widetilde{L_{\beta,+}} = \begin{pmatrix} \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta + 1 & 0 & 0 \\ 0 & \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta + 1 & 0 \\ 0 & 0 & \mu\partial_\xi^2 + (c-2\mu\beta)\partial_\xi + \mu\beta^2 - c\beta \end{pmatrix},$$

since $A_{\text{front}}(\xi) \to 0$ for $\xi \to +\infty$. Applying Fourier transform then yields

$$\sigma(\widetilde{L_{\beta,+}}) = \bigcup_{k\in\mathbb{R}} \{-k^2 + (c-2\beta)ik + \beta^2 - c\beta + 1, -\mu k^2 + (c-2\mu\beta)ik + \mu\beta^2 - c\beta\}.$$

Similarly, using $A_{\text{front}}(\xi) \to 1$ for $\xi \to -\infty$ we obtain

$$\widetilde{L_{\beta,-}} = \begin{pmatrix} \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta - 2 & 0 & 1 \\ 0 & \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta - 2 & \partial_\xi - \beta \\ 0 & 0 & \mu\partial_\xi^2 + (c-2\mu\beta)\partial_\xi + \mu\beta^2 - c\beta \end{pmatrix},$$

where we used that all $w \in \mathcal{D}(L_\beta)$ satisfy $w_C = (\partial_\xi - \beta)w_A$. Noting that $\widetilde{L_{\beta,-}}$ is an upper triangular matrix, we obtain using Fourier transform that

$$\sigma(\widetilde{L_{\beta,+}}) = \bigcup_{k\in\mathbb{R}} \{-k^2 + (c-2\beta)ik + \beta^2 - c\beta - 2, -\mu k^2 + (c-2\mu\beta)ik + \mu\beta^2 - c\beta\}.$$

In particular, this yields that $\mathrm{Re}(\sigma_{\text{ess}}(\widetilde{L_\beta})) \le -\kappa$.

Secondly, we control the point spectrum. Therefore, consider the eigenvalue problem

$$\lambda w = \widetilde{L_\beta} w \tag{30}$$

for $w \in \mathcal{D}(\widetilde{L_\beta})$. We show that this eigenvalue problem (30) has no nontrivial solution if $\mathrm{Re}(\lambda) > -\kappa$. First note that the w_B-equation reads as

$$\lambda w_B = \mu\partial_\xi^2 w_B + (c-2\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta)w_B =: \mathcal{L}_B w_B$$

and is decoupled from the remaining system. In particular, any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\kappa$ is in the resolvent set of $\mathcal{L}_B$ and hence any solution of (30) satisfies $w_B = 0$. Thus, the w_A-equation reads as

$$\lambda w_A = \partial_\xi^2 w_A + (c-2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1)w_A - 3A_{\text{front}}^2 w_A =: \mathcal{L}_A w_A.$$

Using $-3A_{\text{front}}^2 < 0$ and comparison principle, see e.g. [ES00], we find that $\mathrm{Re}(\sigma(\mathcal{L}_A)) \le -\kappa$ and thus, $w_A = 0$. Since all $w \in \mathcal{D}(\widetilde{L_\beta})$ satisfy $w_C = (\partial_\xi - \beta)w_A$, this also implies $w_C = 0$. Hence, (30) has only the trivial solution in $\mathcal{D}(\widetilde{L_\beta})$ if $\mathrm{Re}(\lambda) > -\kappa$. This concludes the proof. □

It remains to analyze the spectrum of L_ψ. Therefore, we proof the following result.

Lemma 4.2. *Let $c > 2$ then there exists a $\beta_\psi^0 > 0$, depending on c, such that for all $\beta_\psi \in (0, \beta_\psi^0)$ there exists a $\gamma_0 > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds that*

$$\mathrm{Re}(\sigma(L_\psi)) < -\kappa_\psi < 0$$

for some $\kappa_\psi = \kappa_\psi(c, \beta_\psi, \gamma_0) > 0$.

Proof. Similar to the analysis of $\widetilde{L_\beta} + P(\gamma)$ we first set $\gamma = 0$. Let $w_\psi \in \mathcal{D}(L_\psi)$ with $\|w_\psi\|_{L^2} = 1$. Then, we estimate

$$\begin{aligned}
\operatorname{Re}\langle L_\psi w_\psi, w_\psi\rangle_{L^2} &= -\|\partial_\xi w_\psi\|_{L^2}^2 + (\beta_\psi^2 - c\beta_\psi) + 2\operatorname{Re}\left\langle (\partial_\xi - \beta_\psi)\left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} w_\psi\right), w_\psi\right\rangle_{L^2} \\
&= -\|\partial_\xi w_\psi\|_{L^2}^2 + (\beta_\psi^2 - c\beta_\psi) + 2\left\langle \left(-\beta_\psi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} + \frac{1}{2}\partial_\xi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}\right) w_\psi, w_\psi\right\rangle_{L^2} \\
&\leq -\|\partial_\xi w_\psi\|_{L^2}^2 + \beta_\psi^2 - c\beta_\psi + \sup\left(-2\beta_\psi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} + \partial_\xi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}\right) \\
&=: -\|\partial_\xi w_\psi\|_{L^2}^2 + \beta_\psi^2 - c\beta_\psi + \sup(f)
\end{aligned}$$

It remains to calculate the supremum of f. Therefore, we note that for $\gamma = 0$, $\partial_\xi A_{\text{front}}/A_{\text{front}}$ is negative and monotonically decreasing, see Proposition A.3. This yields with Proposition A.1 that

$$\sup(f) \leq \beta_\psi(c - \sqrt{c^2 - 4}).$$

Therefore, we obtain that

$$\operatorname{Re}(\sigma(L_\psi)) \leq \beta_\psi^2 - \beta_\psi\sqrt{c^2 - 4}$$

and thus, for $c > 2$, there exists a β_ψ^0 such that $\operatorname{Re}(\sigma(L_\psi)) < -\kappa_\psi$ for all $\beta_\psi \in (0, \beta_\psi^0)$ with $\kappa_\psi = \kappa_\psi(\beta_\psi) > 0$. Then, using perturbation arguments, we obtain the result also for γ close to zero. □

Now, using Lemmas 4.1 and 4.2, (29) and applying [Kat66, Theorem IV.3.17] we obtain the spectral stability of the full weighted operator.

Lemma 4.3. *Let $\mu > 0$ and $c > c_{min}(\mu)$. There exists a $\beta_\psi^0 > 0$ such that for all $\beta > 0$, $\beta_\psi > 0$ satisfying*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) < 0 \\
&\beta_\psi \in (0, \beta_\psi^0)
\end{aligned}$$

exists a $\tilde{\kappa} > 0$ such that there exists a $\gamma_0 > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds

$$\operatorname{Re}(\sigma(L_\beta)) \leq -\frac{\tilde{\kappa}}{2}.$$

Remark 4.4. Using similar arguments, we can also deduce that L_β is a closed operator and in particular sectorial in L^2. □

4.2 Linear stability estimates

After proving the spectral stability of L_β in the previous section, we now focus on the linear stability. In particular, we aim to prove the linear stability estimates (20), (21), which we restate below. Therefore, we fix parameters $\beta > 0$, $\beta_\psi > 0$, $\mu > 0$ and $c > c_{\min}(\mu)$ such that Lemma 4.3 applies. In this case, L_β is a closed, sectorial operator in L^2 and hence, standard semigroup theory implies that L_β generates an analytic semigroup in L^2, see e.g. [EN00], which we denote by e^{tL_β} for all $t \geq 0$. We now prove that the estimates

$$\left\|e^{tL_\beta} w\right\|_{L^2} \lesssim e^{-\kappa t}\|w\|_{L^2}, \tag{31}$$

$$\left\|e^{tL_\beta}\partial_\xi w\right\|_{L^2} \lesssim (1 + t^{-1/2})e^{-\kappa t}\|w\|_{L^2}, \tag{32}$$

hold for some $\kappa > 0$ and all $w \in L^2$, $t > 0$.

For the first estimate, we recall two results from semigroup theory, see e.g. [EN00]. First, we define the growth bound of a strongly continuous semigroup with generator L by

$$\eta_0(L) := \inf\{\eta \in \mathbb{R} : \exists M < \infty \text{ such that } \left\|e^{tL}\right\| \leq Me^{\eta t}\}.$$

Note that this is well-defined for all strongly continuous semigroups and that the growth bound is connected to the spectral radius of e^{tL} by

$$\eta_0(L) = \frac{1}{t}\log(r(e^{tL})),$$

for some fixed $t > 0$, where $r(e^{tL}) := \sup\{|\lambda| : \lambda \in \sigma(e^{tL})\}$, see [EN00, Proposition IV.2.2]. Furthermore, if L is a sectorial operator, then the Spectral Mapping Theorem holds, i.e.

$$\sigma(e^{tL}) \setminus \{0\} = \{e^{\lambda t} : \lambda \in \sigma(L)\},$$

see [EN00, Corollary IV.3.12]. Using these results, we obtain the first linear stability estimate (31) by using that L_β is a sectorial operator in L^2 and that $\mathrm{Re}(\sigma(L_\beta))$ is strictly negative.

We now consider the second linear stability estimate (32). To prove this, we introduce operators T, containing the terms with constant coefficients and the remainder S, such that $L_\beta = T + S$. Next, we introduce the operators S_1, S_2 such that

$$S\partial_\xi w = (\partial_\xi S_1)w + S_2 w$$

for $w \in H^1$. We note that S, S_1 and S_2 are T-bounded – in fact, S_2 is a bounded operator in L^2 – with constants $\tilde{a}, \tilde{b}$, see [Kat66, Chapter IV.1.1] for a definition. The main result leading to (32) is the following resolvent estimate.

Lemma 4.5. *There exists a sector $\mathcal{S}(w_0, \varphi) = \{\lambda \in \mathbb{C} : \arg(\lambda - w_0) \in (\pi - \varphi, \pi + \varphi)\} \subset \mathbb{C}$, $w_0 \in \mathbb{R}$, $\varphi \in (0, \pi)$ such that*

$$\|R(\lambda, L_\beta)\partial_\xi w\|_{L^2} \leq \delta^{-1}\|R(\lambda, T)\partial_\xi\|_{L^2\to L^2}\|w\|_{L^2} + \frac{1-\delta}{\delta^2}\|R(\lambda, T)\|_{L^2\to L^2}\|w\|_{L^2},$$

for all $\lambda \in \mathbb{C} \setminus \mathcal{S}(w_0, \varphi)$ and all $w \in H^1$. Here, $R(\lambda, \mathcal{L}) := (\lambda - \mathcal{L})^{-1}$ is the resolvent map, which is defined on the resolvent set.

Proof. For any chosen $\delta \in (0, 1)$ we choose a sector $\mathcal{S}(w_0, \varphi)$ such that

$$\tilde{a}\|R(\lambda, T)\|_{L^2\to L^2} + \tilde{b}\|TR(\lambda, T)\|_{L^2\to L^2} < 1 - \delta,$$

which in particular yields

$$\|JR(\lambda, T)\|_{L^2\to L^2} \leq (1 - \delta)$$

for $J \in \{S, S_1, S_2\}$, since J is T-bounded. This is possible using that T is a constant coefficient, second-order operator and thus $R(\lambda, T)$ and $TR(\lambda, T)$ can be estimates using Fourier transform. Hence, following the proof of [Kat66, Theorem IV.3.17], we have that for all $\lambda \in \mathbb{C}\setminus\mathcal{S}(w_0, \varphi)$ holds that $\|SR(\lambda, T)\|_{L^2\to L^2} < 1-\delta$ and that

$$R(\lambda, L_\beta) = R(\lambda, T)(1 - SR(\lambda, T))^{-1} = R(\lambda, T)\sum_{n=0}^{\infty}(SR(\lambda, T))^n.$$

Furthermore, we have that $\partial_\xi S_1$ is T-bounded, since it is a second order differential operator with smooth coefficients. This yields that $\|\partial_\xi S_1 R(\lambda, T)\|_{L^2\to L^2} < K < \infty$ and especially,

$$\|\partial_\xi(S_1 R(\lambda, T))^n\|_{L^2\to L^2} \leq K\|S_1 R(\lambda, T)\|^{n-1}_{L^2\to L^2} < K(1 - \delta)^{n-1}$$

for all $n \geq 1$. Hence, the series $\sum_{n=1}^{\infty}\partial_\xi(S_1 R(\lambda, T))^n$ is well-definied and a bounded operator mapping $L^2 \to L^2$.

Now let $w \in H^1$ be arbitrary. Then, in particular $\sum_{n=1}^{N}(S_1R(\lambda,T))^n w$ converges in H^1 for $N \to \infty$ and thus, since $\partial_\xi : H^1 \to L^2$ is closed this yields

$$\partial_\xi \sum_{n=0}^{\infty}(S_1R(\lambda,T))^n w = \sum_{n=0}^{\infty}\partial_\xi(S_1R(\lambda,T))^n w.$$

Hence, it holds that

$$\begin{aligned}
R(\lambda,L_\beta)\partial_\xi w &= R(\lambda,T)\sum_{n=0}^{\infty}(SR(\lambda,T))^n\partial_\xi w \\
&= R(\lambda,T)\sum_{n=0}^{\infty}\left[\partial_\xi(S_1R(\lambda,T))^n w + \sum_{j=0}^{n-1}(SR(\lambda,T))^{n-1-j}S_2R(\lambda,T)(S_1R(\lambda,T))^j w\right] \\
&= R(\lambda,T)\partial_\xi\sum_{n=0}^{\infty}(S_1R(\lambda,T))^n w + R(\lambda,T)\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}(SR(\lambda,T))^{n-1-j}S_2R(\lambda,T)(S_1R(\lambda,T))^j w,
\end{aligned}$$

where we used the definition of S_1, S_2. Using the choice of $\mathcal{S}(w_0,\varphi)$ and the fact that $JR(\lambda,T)$, $J \in \{S, S_1, S_2\}$ are bounded operators on $L^2 \to L^2$, we can estimate that

$$\begin{aligned}
\sum_{j=0}^{n-1}\left\|(SR(\lambda,T))^{n-1-j}S_2R(\lambda,T)(S_1R(\lambda,T))^j\right\|_{L^2\to L^2} &\le n(\tilde{a}\left\|R(\lambda,T)\right\|_{L^2\to L^2} + \tilde{b}\left\|TR(\lambda,T)\right\|_{L^2\to L^2})^n \\
&< n(1-\delta)^n,
\end{aligned}$$

which finally yields that

$$\left\|R(\lambda,L_\beta)\partial_\xi w\right\|_{L^2} \le \delta^{-1}\left\|R(\lambda,T)\partial_\xi\right\|_{L^2\to L^2}\left\|w\right\|_{L^2} + \frac{1-\delta}{\delta^2}\left\|R(\lambda,T)\right\|_{L^2\to L^2}\left\|w\right\|_{L^2}$$

for all $w \in H^1$. This concludes the proof. □

Note that the above result is not enough to proceed yet since $\mathcal{S}(w_0,\varphi)$, in general, contains a subset of $\mathbb{C}^+$. Therefore, we fix $\gamma \in (-\gamma_0,\gamma_0)$ with γ_0 from Lemma 4.3 with corresponding $\tilde{\kappa} > 0$. Then, we show that for all $\lambda \in \Omega := \mathcal{S}(w_0,\varphi) \cap \{\mathrm{Re}(\lambda) > -\tilde{\kappa}/4\}$ there exist a constant $K < \infty$ such that

$$\left\|R(\lambda,L_\beta)\partial_\xi w\right\|_{L^2} \le K\left\|w\right\|_{L^2}. \tag{33}$$

Therefore, we use that for $\lambda_1,\lambda_2 \in \rho(L_\beta)$ with $|\lambda_1-\lambda_2|$ small enough, the resolent $R(\lambda_2,L_\beta)$ can be expanded into a Neumann series such that

$$R(\lambda_2,L\beta) = \sum_{n=0}^{\infty}(\lambda_2-\lambda_1)^n R(\lambda_1,L_\beta)^{n+1}$$

holds, where the sum converges absolutely. Recalling that $\mathrm{Re}(\sigma(L_\beta)) < -\tilde{\kappa}/2$ using Lemma 4.3, we can bound $R(\lambda,L_\beta)$ uniformly on Ω since it is bounded away from the spectrum of L_β. Furthermore, using the fact that Ω is compact, we can find a finite set of balls $\mathcal{B}_l$, which covers Ω. Then, by choosing the $\mathcal{B}_l$ small enough, we can estimate

$$\left\|R(\lambda_2,L_\beta)\partial_\xi w\right\|_{L^2} \le \left\|R(\lambda_1,L_\beta)\partial_\xi w\right\|_{L^2}\sum_{n=0}^{\infty}|\lambda_2-\lambda_1|^n\left\|R(\lambda_1,L_\beta)\right\|_{L^2\to L^2}^n$$

for all $\lambda_1,\lambda_2 \in \mathcal{B}_l$ for some l. Then using Lemma 4.5, we obtain (33) for all $\lambda \in \Omega$.

Now, using the above Lemma 4.5 and (33), we can derive (32) by noting that L_β is a sectorial operator and thus, the semigroup is defined by

$$e^{tL_\beta} w = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} R(\lambda, L_\beta) w \, d\lambda,$$

for a curve Γ in the resolvent set of L_β going from $\infty e^{i(\pi/2+\tilde{\delta})}$ to $\infty e^{-i(\pi/2+\tilde{\delta})}$ for some $\tilde{\delta} \in (0, \vartheta)$, with ϑ from Lemma 4.5. Furthermore, we choose Γ such that $\mathrm{Re}(\Gamma) \leq -\tilde{\kappa}/4$. In particular, for all $\lambda \in \Gamma + \tilde{\kappa}/4$ we have the estimate

$$\|R(\lambda, T)\partial_\xi\|_{L^2 \to L^2} \lesssim |\lambda|^{-1/2} \text{ and } \|R(\lambda, T)\|_{L^2 \to L^2} \lesssim |\lambda|^{-1},$$

which can be checked using Fourier analysis, since T is a constant coefficient operator and explicitly given. Using the resolvent estimate provided in Lemma 4.5 and the fact that T generates a strongly continuous semigroup yields

$$\left\|e^{tL_\beta}\partial_\xi w\right\|_{L^2} \lesssim e^{-(\tilde{\kappa}/4)t} \int_{\Gamma+\tilde{\kappa}/4} |e^{\lambda t}| (|\lambda|^{-1/2} + |\lambda|^{-1}) \, d\lambda \, \|w\|_{L^2} \lesssim (1 + t^{-1/2}) e^{-\tilde{\kappa} t/4} \, \|w\|_{L^2} \, ,$$

which proves the linear stability estimate (32) for all $w \in H^1$. Furthermore, using that H^1 is dense in L^2 the estimate also holds for $w \in L^2$. This concludes the proof of Lemma 3.1.

Remark 4.6. We hightlight that it is sufficient to show (33) on Ω since the crucial part of the semigroup estimate comes from the decay of the resolvent for $|\,\mathrm{Re}(\lambda)| \to \infty$. □

5 Discussion

In this paper, we have proven the nonlinear stability of traveling front solutions with velocity $c > c_{\min}(\mu)$ of a Ginzburg-Landau equation with an additional conservation law with coupling parameter γ close to zero. This nonlinear stability holds with respect to localized perturbations, which exhibit spatial exponential decay ahead of the front.

We now discuss related and open questions.

Stability of slower fronts. The results in this paper are restricted to sufficiently fast invading fronts, i.e. $c > c_{\min}(\mu)$. Therefore, a natural question is if a similar stability result can also be established for slower fronts. For this discussion, we consider the cases $\mu \in (0, 2)$ and $\mu > 2$ separately.

First, we consider the case $\mu \in (0, 2)$, in which $c_{\min}(\mu) = 2$. For $c < c_{\min}(\mu)$, the weighted operator L_β has spectrum with positive real part, which leads to instability of the corresponding invading front. However, for $c = c_{\min}(\mu)$ we can chose weights β and β_ψ such that the weighted operator L_β has spectrum up to the imaginary axis, although this cannot be improved. In the scalar case such as a real Ginzburg-Landau equation and the Fisher-KPP equation, the stability of these critical fronts is well understood, see e.g. [Kir92, Gal94, FH19a]. The proof relies on obtaining sufficiently good polynomial decay estimates for the semigroup in appropriately chosen weighted spaces. There are recent advances to prove the stability of critical fronts in the system case using pointwise estimates, see [FH19b]. However, these techniques require some structure of the equation which allows to exclude the presence of unstable point spectrum and in particular the presence of an eigenvalue at $\lambda = 0$ in an appropriately weighted space. Therefore, to apply these methods to the fronts studied in this paper, a more precise analysis of the decay rates of the fronts $(A_{\text{front}}, B_{\text{front}})$ and of the weighted operator L_β is necessary.

Second, we consider the case $\mu > 2$, in which $c_{\min}(\mu) > 2$. Similar to the first case, we also find that for $c = c_{\min}(\mu)$, the weighted operator L_β has spectrum up to the imaginary axis for β and β_ψ chosen optimally. However, by equipping the r_A- and r_B-components with different weights it is possible to stabilize the spectrum as long as $c \geq 2$. In fact, it is possible to obtain (linear) exponential stability for $c > 2$. Although this suggests that the stability result can be extended to $c > 2$, even if $\mu > 2$, a proof is an open question

since using different weights necessarily leads to unbounded terms in the weighted nonlinearity and thus, we cannot proceed as done in this paper.

Stability of modulating fronts As discussed in the introduction, the motivation to study the stability of invading fronts in (1) is that this system appears as a generic amplitude equation in pattern forming systems admitting a conservation law structure, see [Zim14]. In particular, in [Hil20] it is shown that the invading fronts studied here approximate the amplitude of modulating fronts, see Figure 3. Since the invading fronts are nonlinearly stable provided they are sufficiently fast, a natural question is if this can be extended to the full modulating fronts. The main difference, compared to the analysis presented in this paper is the fact that modulating fronts model an invasion by a periodic state instead of a homogeneous one. Therefore, we cannot analyze the invading state using Fourier analysis, but instead have to use Bloch wave analysis as in [ES00, ES02]. For more details, we refer to the discussion in [Hil20, Section 5].

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A Properties of front solutions

The aim of this section is to provide some important analytical properties of the front solutions of (1).

A.1 Decay properties of front solutions

We study the asymptotic spatial behavior of the front, i.e. how fast the front approaches its asymptotic rest state for $\xi \to +\infty$. Therefore, recall that the front solutions $A_{\text{front}}, B_{\text{front}}$ are real stationary solutions of the spatial dynamics formulation of (1) in a co-moving frame, i.e. for $c > 2$ we have

$$\begin{aligned} 0 &= \partial_\xi^2 A_{\text{front}} + c\partial_\xi A_{\text{front}} + A_{\text{front}} + A_{\text{front}} B_{\text{front}} - A_{\text{front}}^3, \\ 0 &= \mu\partial_\xi^2 B_{\text{front}} + c\partial_\xi B_{\text{front}} + \gamma\partial_\xi^2 (A_{\text{front}}^2). \end{aligned}$$

After integrating the B_{front} equation once, we write this system as a first order system in ξ we obtain

$$\partial_\xi A_{\text{front}} = \widetilde{A_{\text{front}}}, \tag{34a}$$

$$\partial_\xi \widetilde{A_{\text{front}}} = -c\widetilde{A_{\text{front}}} - A_{\text{front}} - A_{\text{front}} B_{\text{front}} + A_{\text{front}}^3, \tag{34b}$$

$$\partial_\xi B_{\text{front}} = -\frac{1}{\mu}(cB_{\text{front}} + 2\gamma\widetilde{A_{\text{front}}} A_{\text{front}}). \tag{34c}$$

To calculate the decay rate to zero for $\xi \to -\infty$, we linearize about $(0, 0, 0)$ and find the linear operator

$$\mathcal{L}_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -c & 0 \\ 0 & 0 & -\frac{c}{\mu} \end{pmatrix}.$$

The eigenvalues of $\mathcal{L}_0$ are given by

$$\lambda_\pm = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4} \text{ and } \lambda_3 = -\frac{c}{\mu}.$$

Note in particular, that for $c > 2$ all eigenvalues are simple and thus we have the following result.

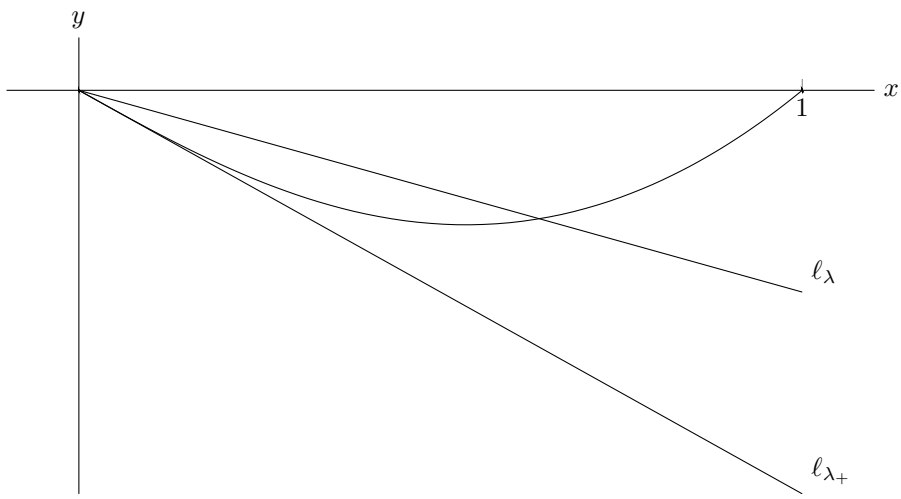

Figure 7: Phase plane depicting the heteroclinic orbit as well as ℓ_{λ_+} and ℓ_λ for $\lambda \in (\lambda_+, 0)$.

Proposition A.1. *Let $c > 2$, $\mu > 0$. Then it holds that*

$$A_{front}(\xi) = \mathcal{O}(e^{\lambda_+ \xi}),\ \partial_\xi A_{front} = \mathcal{O}(\lambda_+ e^{\lambda_+ \xi}),\ \text{and}\ B_{front} = \mathcal{O}(e^{\lambda_3 \xi})$$

for $\xi \to +\infty$.

Remark A.2. Note that the condition on the weight β in Lemma 4.3 is equivalent to $\beta \in (-\lambda_+, -\lambda_-)$ and $\beta \in (0, -\lambda_3)$. Hence we require in particular that the perturbation decays faster than the front for $\xi \to \infty$. □

A.2 Properties of $\partial_\xi A_{\text{front}}/A_{\text{front}}$

We now prove the results for the logarithmic derivative of A_{front}, which are needed in the spectral analysis of L_ψ in Lemma 4.2.

Proposition A.3. *Let $c > 0$ and $\gamma = 0$. Then, $f_\psi := \partial_\xi A_{front}/A_{front}$ is negative, monotonically decreasing and satisfies*

$$\lim_{\xi \to \infty} f(\xi) = \lambda_+ = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 - 4}.$$

Proof. We first note that the asymptotic limiting property is a well-known result for fronts of the Ginzburg-Landau equation, see e.g. [Sat76]. Furthermore, since A_{front} is monotonically decreasing, f_ψ is negative. Thus, it remains to prove that f_ψ is monotonically decreasing. For that we use phase plane analysis. Therefore, recall that for $\gamma = 0$, A_{front} satisfies $\partial_\xi^2 A_{\text{front}} = -c\partial_\xi A_{\text{front}} - A_{\text{front}} + A_{\text{front}}^3$. Writing this as a first order system we obtain

$$\partial_\xi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -cy - x + x^3 \end{pmatrix},$$

which exhibits a heterclinic orbit (x, y) connecting $(1, 0)$ to $(0, 0)$. Next, for $\lambda \in [\lambda_+, 0)$ we define $\ell_\lambda := \lambda x$. We now show that $f_\psi = y/x$ intersects each ℓ_λ exactly once for $\lambda \in (\lambda_+, 0)$, see Figure 7.

The idea of the proof is to show that for all $\lambda \in (\lambda_+, 0)$ exists exactly one $x_\lambda \in (0, 1)$ such that

$$\frac{dy}{dx}|_{y=\ell_\lambda} - \lambda = 0, \tag{35}$$

i.e. the slope of the vector field coincides with the slope of ℓ_λ. Since $f_\psi \to \lambda_+$ for $\xi \to \infty$ and $f_\psi \to 0$ for $\xi \to -\infty$, f_ψ has to intersect each ℓ_λ at least once. Now suppose that f_ψ is non-monotone. Then, there exists a $\lambda \in (\lambda_+, 0)$ such that f_ψ intersects with ℓ_λ at least three times. However, this contradicts the fact that (35) is only satisfies for exactly one $x_\lambda \in (0,1)$. Hence, f_ψ is monotonically decreasing.

It remains to show that (35) is satisfied for exactly one $x_\lambda \in (0,1)$. This is a direct calculation.

$$\frac{dy}{dx}|_{y=\ell_\lambda} - \lambda = -c - \frac{1-x^2}{\lambda} - \lambda = 0 \quad \Longleftrightarrow \quad x_\lambda = \sqrt{\lambda^2 + c\lambda + 1},$$

which is well-defined since $0 < \lambda^2 + c\lambda + 1 \leq 1$ for all $\lambda \in (\lambda_+, 0)$. □

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