Modulationsgleichungen Vorlesung 14 Mo, 30.05.2016

Wir können die Energieabschätzungen beweisen. Es gilt:

$$\begin{aligned}\frac{d}{dt}E_\ell = \sum_{j_1\in\{\pm 1\}}\Bigg(&\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_t\partial_x^\ell R_{j_1}\,dx + \varepsilon\sum_{j_2\in\{\pm 1\}}\bigg(\int_{\mathbb{R}}\partial_t\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi, R_{j_2})\,dx\\ &+\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi,\partial_t R_{j_2})\,dx + \int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x N_{j_1j_2}(\partial_t\psi, R_{j_2})\,dx\bigg)\Bigg).\end{aligned}$$

Mit Hilfe der Fehlergleichungen folgt:

$$\begin{aligned}\frac{d}{dt}E_\ell = &\sum_{j_1\in\{\pm 1\}}\bigg(j_1\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,i\omega\,\partial_x^\ell R_{j_1}\,dx + \int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\varepsilon^{-5/2}\,\partial_x^\ell \mathrm{Res}_{u_{j_1}}(\varepsilon\Psi)\,dx\bigg)\\ &+\varepsilon\sum_{j_1,j_2\in\{\pm 1\}}\bigg(\underline{j_1\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,i\varrho\,\partial_x^\ell(\psi R_{j_2})\,dx} + \underline{j_1\int_{\mathbb{R}} i\omega\,\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi, R_{j_2})\,dx}\\ &\quad+\underline{j_2\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi, i\omega R_{j_2})\,dx} - \underline{\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(i\omega\psi, R_{j_2})\,dx}\\ &\quad+\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\partial_t\psi - i\omega\psi, R_{j_2})\,dx - \int_{\mathbb{R}}\varepsilon^{-5/2}\,\partial_x^{\ell+1}\mathrm{Res}_{u_{j_1}}(\varepsilon\Psi)\,\partial_x^{\ell-1}N_{j_1j_2}(\psi, R_{j_2})\,dx\\ &\quad+\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi,\varepsilon^{-5/2}\mathrm{Res}_{u_{j_2}}(\varepsilon\Psi))\,dx\bigg)\\ &+\varepsilon^2\sum_{j_1,j_2,j_3\in\{\pm 1\}}\bigg(j_1\int_{\mathbb{R}} i\varrho\,\partial_x^\ell(\tilde\psi R_{j_3})\,\partial_x^\ell N_{j_1j_2}(\psi, R_{j_2})\,dx + j_2\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,\partial_x^\ell N_{j_1j_2}(\psi, i\varrho(\tilde\psi R_{j_3}))\,dx\bigg)\\ &+\varepsilon^{5/2}\sum_{j_1,j_2,j_3\in\{\pm 1\}}\frac{j_1}{2}\int_{\mathbb{R}}\partial_x^\ell R_{j_1}\,i\varrho\,\partial_x^\ell\big(R_{j_3}R_{j_2}\big)\,dx\end{aligned}$$

mit $\tilde\psi = \psi + \frac{\varepsilon^{3/2}}{2}(R_1 + R_{-1})$. Da $i\omega$ schiefsymmetrisch ist, verschwindet das erste Integral. Aufgrund der NFT-Eigenschaft von $N_{j_1j_2}$ aus Lemma 1.2.6 b.) verschwindet die Summe aus dem dritten, vierten, fünften und sechsten Integral.

Aufgrund der obigen Abschätzungen für das Residuum und für $\partial_t \psi_{\pm 1} - i\omega \psi_{\pm 1}$, der Regularitätseigenschaften von $N_{j_1 j_2}$ aus Lemma 1.2.6 a.), der ersten Identität aus Lemma 1.2.8 und Korollar 1.2.7 können das zweite, siebte, achte und neunte Integral durch $C\varepsilon^2(\mathcal{E}_s + 1)$ mit $C > 0$ beschränkt werden. Also bleibt

$$\frac{d}{dt} E_\ell = \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \Big(j_1 \int_{\mathbb{R}} i g \partial_x^\ell (\tilde{\psi} R_{j_3}) \partial_x^\ell N_{j_1 j_1}(\psi, R_{j_1})\, dx$$

$$+ j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} \partial_x^\ell N_{j_1 j_1}(\psi, i g(\tilde{\psi} R_{j_3}))\, dx + j_1 \int_{\mathbb{R}} i g\, \partial_x^\ell (\tilde{\psi} R_{j_3}) \partial_x^\ell N_{j_1 -j_1}(\psi, R_{-j_1})\, dx$$

$$- j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} \partial_x^\ell N_{j_1 -j_1}(\psi, i g(\tilde{\psi} R_{j_3}))\, dx$$

$$+ \varepsilon^{5/2} \sum_{j_1, j_2, j_3 \in \{\pm 1\}} \frac{j_1}{2} \int_{\mathbb{R}} \partial_x^\ell R_{j_1}\, i g\, \partial_x^\ell (R_{j_3} R_{j_2})\, dx \quad + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + 1)$$

$$=: \sum_{j=1}^{5} I_j \quad + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + 1).$$

Es gilt:

$$I_1 + I_2 \overset{\text{Leibniz-Regel}}{=} \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \Big(j_1 \int_{\mathbb{R}} i g\, \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{j_1 j_1}(\psi, \partial_x^\ell R_{j_1})\, dx$$

$$+ \ell j_1 \int_{\mathbb{R}} i g\, \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{j_1 j_1}(\partial_x \psi, \partial_x^{\ell-1} R_{j_1})\, dx$$

$$+ j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} N_{j_1 j_1}(\psi, i g\, \partial_x^\ell (\tilde{\psi} R_{j_3}))\, dx$$

$$+ \ell j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} N_{j_1 j_1}(\partial_x \psi, i g\, \partial_x^{\ell-1} (\tilde{\psi} R_{j_3}))\, dx$$

$$+ \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big).$$

Mit Hilfe von Lemma 1.2.6 c.) folgt

$$I_1 + I_2 = \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \Big(j_1 \int_{\mathbb{R}} i g\, \partial_x^\ell (\tilde{\psi} R_{j_3})\, S_{j_1 j_1}(\partial_x \psi, \partial_x^\ell R_{j_1})\, dx$$

$$+ 2\ell j_1 \int_{\mathbb{R}} i g\, \partial_x^\ell (\tilde{\psi} R_{j_3})\, N_{j_1 j_1}(\partial_x \psi, \partial_x^{\ell-1} R_{j_1})\, dx \Big) + \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big)$$

Lemma 1.2.6 a.), c.) $\wedge$ (9)

$$\overset{\downarrow}{=} -(2\ell+1)\, \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \int_{\mathbb{R}} G_{j_1 j_1}\, \partial_x \psi\, \tilde{\tilde{\psi}}\, \partial_x^\ell R_{j_1}\, \partial_x^{\ell+1} R_{j_3}\, dx + \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big)$$

mit $\tilde{\tilde{\psi}} = \psi + \varepsilon^{3/2}(R_1 + R_{-1})$. Wegen Lemma 1.2.8 und

$$(\hat{G}_{-1-1} - \hat{G}_{11})(k) = \frac{2ik\,\chi(k)}{\omega^2(k) - k^2} = 2ik\,\chi(k)$$

folgt

$$I_1 + I_2 = -\frac{2\ell+1}{2}\, \varepsilon^2 \int_{\mathbb{R}} (G_{-1-1} - G_{11})\, \partial_x \psi\, \partial_x^\ell (R_1 + R_{-1})\, \partial_x^{\ell+1} (R_1 - R_{-1})\, dx$$

$$+ \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big)$$

$$= -(2\ell+1)\, \varepsilon^2 \int_{\mathbb{R}} \partial_x^2 \psi\, \tilde{\tilde{\psi}}\, \partial_x^\ell (R_1 + R_{-1})\, \partial_x^{\ell+1} (R_1 - R_{-1})\, dx$$

$$+ \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big).$$

Wegen $\partial_t (R_1 + R_{-1}) = i\omega (R_1 - R_{-1})$ und (7) folgt

$$I_1 + I_2 = -(2\ell+1)\, \varepsilon^2 \int \partial_x^2 \psi\, \tilde{\tilde{\psi}}\, \partial_x^\ell (R_1 + R_{-1})\, \partial_t \partial_x^\ell (R_1 + R_{-1})\, dx$$

$$+ \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big)$$

$$= -\frac{2\ell+1}{2}\, \varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}} \partial_x^2 \psi \big(\psi + \varepsilon^{3/2}(R_1 + R_{-1})\big) \big(\partial_x^\ell (R_1 + R_{-1})\big)^2 dx$$

$$+ \varepsilon^2 \mathcal{O}\big(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}\big).$$

Analog ergibt sich

$$I_3 + I_4 = \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \Big(j_1 \int_{\mathbb{R}} i\varrho \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{j_1 -j_1}(\psi, \partial_x^\ell R_{-j_1}) \, dx$$

$$- j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} N_{j_1 -j_1}(\psi, i\varrho \partial_x^\ell (\tilde{\psi} R_{j_3})) \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2})$$

$$= \varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} \Big(j_1 \int_{\mathbb{R}} i\varrho \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{j_1 -j_1}(\psi, \partial_x^\ell R_{-j_1}) \, dx$$

$$- j_1 \int_{\mathbb{R}} i\varrho \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{-j_1 j_1}(\psi, \partial_x^\ell R_{j_1}) \, dx$$

$$+ j_1 \int_{\mathbb{R}} \partial_x^\ell (\tilde{\psi} R_{j_3}) i\varrho S_{-j_1 j_1}(\partial_x \psi, \partial_x^\ell R_{j_1}) \, dx \Big) + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}).$$

$$= -\varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} 2 j_1 \int_{\mathbb{R}} i\varrho \, \partial_x^\ell (\tilde{\psi} R_{j_3}) N_{-j_1 j_1}(\psi, \partial_x^\ell R_{j_1}) \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2})$$

$$= -\varepsilon^2 \sum_{j_1, j_3 \in \{\pm 1\}} j_2 \int_{\mathbb{R}} \psi \tilde{\tilde{\psi}} \partial_x^\ell R_{j_1} \partial_x^{\ell+1} R_{j_3} \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2})$$

$$= \varepsilon^2 \int_{\mathbb{R}} \psi \tilde{\tilde{\psi}} \partial_x^\ell (R_1 + R_{-1}) \partial_x^{\ell+1} (R_1 - R_{-1}) \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2})$$

$$= \varepsilon^2 \int_{\mathbb{R}} \psi \tilde{\tilde{\psi}} \partial_x^\ell (R_1 + R_{-1}) \partial_t \partial_x^\ell (R_1 + R_{-1}) \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2})$$

$$= \frac{1}{2} \varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}} \psi (\psi + \varepsilon^{3/2}(R_1 + R_{-1}))(\partial_x^\ell (R_1 + R_{-1}))^2 \, dx + \varepsilon^2 \mathcal{O}(\mathcal{E}_s + \varepsilon^{3/2} \mathcal{E}_s^{3/2}).$$

$$I_5 = \frac{1}{2} \varepsilon^{5/2} \sum_{j_1, j_2, j_3 \in \{\pm 1\}} j_1 \int_{\mathbb{R}} \partial_x^\ell R_{j_1} \partial_x^{\ell+1} (R_{j_3} R_{j_2}) \, dx + \varepsilon^{5/2} \mathcal{O}(\mathcal{E}_s^{3/2})$$

$$= \varepsilon^{5/2} \sum_{j_1, j_2, j_3 \in \{\pm 1\}} j_1 \int_{\mathbb{R}} R_{j_3} \partial_x^{\ell} R_{j_1} \partial_x^{\ell+1} R_{j_2} \, dx + \varepsilon^{5/2} \mathcal{O}(\mathcal{E}_s^{3/2})$$

$$= -\varepsilon^{5/2} \int_{\mathbb{R}} (R_1 + R_{-1}) \partial_x^{\ell} (R_1 + R_{-1}) \partial_x^{\ell+1} (R_1 - R_{-1}) \, dx + \varepsilon^{5/2} \mathcal{O}(\mathcal{E}_s^{3/2})$$

$$= -\varepsilon^{5/2} \int_{\mathbb{R}} (R_1 + R_{-1}) \partial_x^{\ell} (R_1 + R_{-1}) \partial_t \partial_x^{\ell} (R_1 + R_{-1}) \, dx + \varepsilon^{5/2} \mathcal{O}(\mathcal{E}_s^{3/2})$$

$$= -\frac{1}{2} \varepsilon^{5/2} \frac{d}{dt} \int_{\mathbb{R}} (R_1 + R_{-1}) \left(\partial_x^{\ell} (R_1 + R_{-1}) \right)^2 dx + \varepsilon^{5/2} \mathcal{O}(\mathcal{E}_s^{3/2}).$$

Definieren wir nun die modifizierte Energie $\tilde{\mathcal{E}}_s = \mathcal{E}_s + \frac{1}{2} \varepsilon^2 \sum_{\ell=1}^{s} h_\ell$ mit

$$h_\ell = \int_{\mathbb{R}} \Big(\big((2\ell+1) \partial_x^2 \psi - \psi \big) \big(\psi + \varepsilon^{3/2} (R_1 + R_{-1}) \big) + \varepsilon^{1/2} (R_1 + R_{-1}) \Big) \cdot \left(\partial_x^{\ell} (R_1 + R_{-1}) \right)^2 dx$$

dann folgt

$$\frac{d}{dt} \tilde{\mathcal{E}}_s \underset{\substack{\uparrow \\ \text{[eine multiplikative} \\ \text{Konstante]}}}{\lesssim} \varepsilon^2 \left(\tilde{\mathcal{E}}_s + \varepsilon^{1/2} \tilde{\mathcal{E}}^{3/2} + 1 \right).$$

Gronwall
$$\Longrightarrow \sup_{t \in [0, T_0/\varepsilon^2]} \tilde{\mathcal{E}}_s = \mathcal{O}(1).$$

Wegen $\| R_1 + R_{-1} \|_{H^s} \lesssim \sqrt{\tilde{\mathcal{E}}_s}$ für hinreichend kleines $\varepsilon > 0$ und Abschätzung (3) folgt Theorem 1.2.5. $\square$