ON THE MATHEMATICAL JUSTIFICATION OF REDUCED MODELS FOR WATER WAVES

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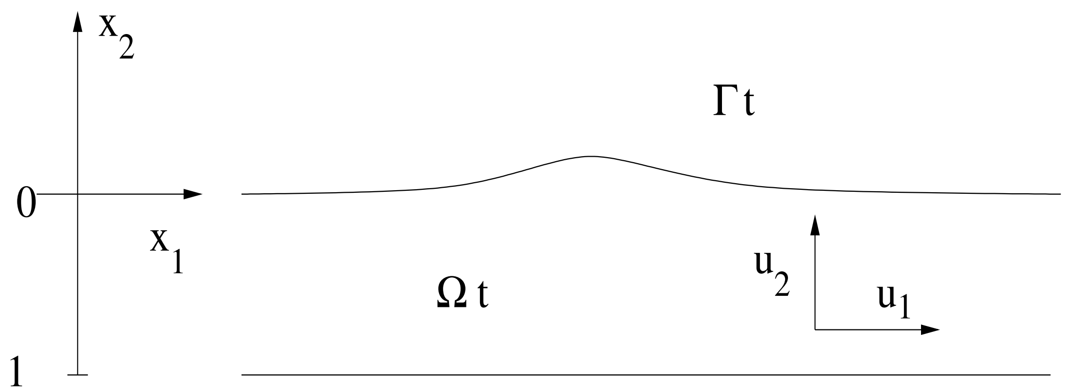
1. Introduction

- Many mathematical models for hydrodynamic problems or for pattern forming systems are so complicated that a qualitative understanding of the full models does not seem within reach for the near future.
- Goal: Approximation of such models in various parameter regimes by appropriate reduced models whose qualitative properties are more easily accessible and mathematically rigorous justifications of these approximations.
- We will present as a typical example the approximation of the two-dimensional water wave equations by the Korteweg-de Vries equation and the Nonlinear Schrödinger equation and discuss its rigorous justification by estimates of the approximation errors in the typical length and time scales.

2. The Two-dimensional Water Wave Problem

- First, we formulate the 2-d water wave problem in Eulerian coordinates:



- *Law of motion* for the velocity field $V = (u_1, u_2)$ of an incompressible, inviscid fluid in an infinitely long canal of finite depth under the influence of gravity:

$$V_t + (V \cdot \nabla)V = -\nabla p - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{in } \Omega(t), \tag{1}$$

$$\nabla \cdot V = 0 \qquad \text{in } \Omega(t) \tag{2}$$

(incompressible Euler equations)

- *Boundary conditions:*
  1. Particles on the free top surface $\Gamma(t) = \eta(x_1, t)$ remain surface particles:

$$\eta_t = V \cdot \begin{pmatrix} -\eta_{x_1} \\ 1 \end{pmatrix} \qquad \text{at } \Gamma(t), \tag{3}$$

  2. Laplace–Young condition for the pressure p:

$$p = -b\kappa \qquad \text{at } \Gamma(t), \tag{4}$$

     b : Bond number (proportional to the strength of the surface tension),
     κ : curvature,

  3. Impermeable bottom B:

$$u_2 = 0 \qquad \text{at } B. \tag{5}$$

- From now on we additionally assume

$$\nabla \times V = 0 \qquad \text{in } \Omega(t). \tag{6}$$

- Then there exists a harmonic velocity potential ϕ and an operator $\mathscr{K} = \mathscr{K}(\eta)$ s.t.

$$V = \nabla\phi \qquad \text{and} \qquad \phi_y = \mathscr{K}\phi_x, \tag{7}$$

where $x = x_1$, $y = x_2$.

- Using (7), the system (1)–(6) can be reduced to

$$\eta_t = \mathscr{K} u_1 - u_1\eta_x \qquad \text{at } \Gamma(t), \tag{8}$$

$$(u_1)_t = -\eta_x - \tfrac{1}{2}((u_1)^2 + (\mathscr{K} u_1)^2)_x + b\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_{xx} \qquad \text{at } \Gamma(t). \tag{9}$$

(Zakharov)

- Alternative formulations of the 2-d water wave model:

1. Using Lagrangian coordinates:

$$\Gamma(t) = \{(\tilde{X}_1(\alpha,t), \tilde{X}_2(\alpha,t)) = (\alpha + X_1(\alpha,t), X_2(\alpha,t)) \,|\, \alpha \in \mathbb{R}\}. \tag{10}$$

2. Arc length formulation:

– Parametrization of the free surface $\Gamma(t)$ by arc length,

– Decomposition of the velocity field on $\Gamma(t)$ in its tangential and its normal component,

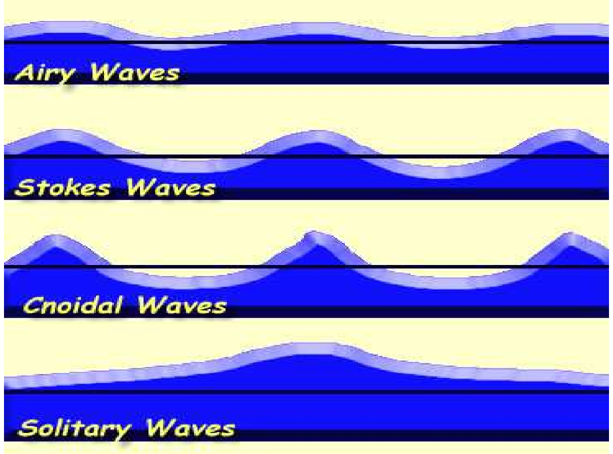
– Analyzing the evolution of the tangent angles on $\Gamma(t)$ instead of the evolution of the tangent slopes on $\Gamma(t)$.

3. Using conformal coordinates.

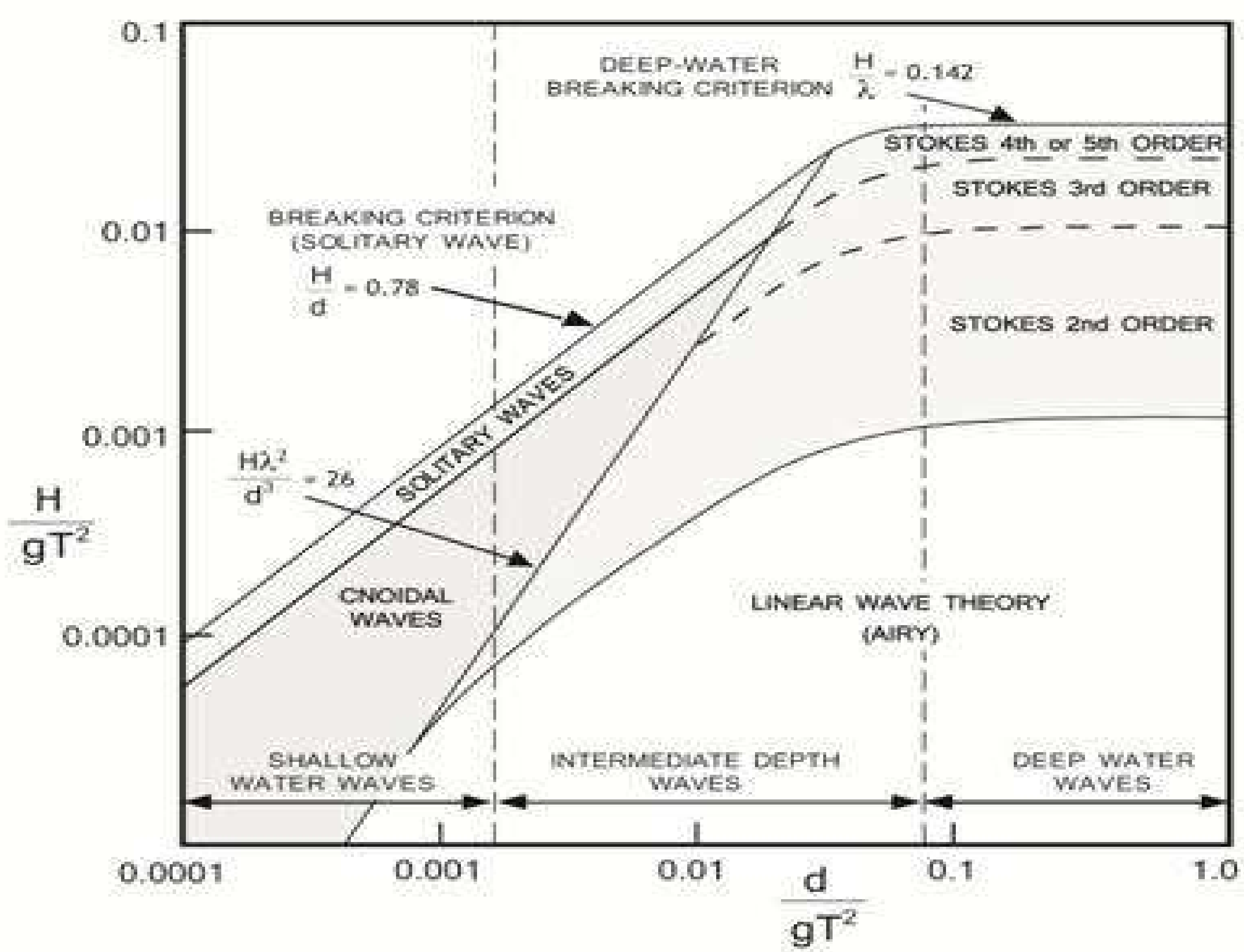
4. Coordinate independent formulation using abstract differential geometric concepts.

- There are several proofs of local existence and uniqueness of solutions to the water wave equations:
  - in Lagrangian coordinates (e.g. Yosihara, Craig, Wu, Schneider–Wayne),
  - in Eulerian coordinates (e.g. Iguchi, Lannes),
  - in the arc length formulation (Ambrose–Masmoudi),
  - in the coordinate independent formulation (Shatah–Zeng),
  - in conformal coordinates (Hunter–Ifrim–Tartaru).
- Global or almost global existence results for sufficiently small initial data are only known in the case of infinite depth of water:
  - for the 2-d water wave equations (Wu, Alazard–Delort, Ionescu–Pusateri, Hunter–Ifrim–Tartaru),
  - for the 3-d water wave equations (Germain–Masmoudi–Shatah, Wu).

Typical profiles of water waves:



- Different profiles in different parameter regimes (Le Méhauté, 1976)



3. The Korteweg-de Vries Approximation

- We consider small and slow modulations of the trivial solution $\eta = u_1 = 0$.
- Inserting the long–wave ansatz

$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix}(x,t) = \varepsilon^2 A\left(\varepsilon(x \pm t), \varepsilon^3 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathscr{O}(\varepsilon^3) \qquad (\varepsilon \ll 1)$$

in (8)–(9) yields at leading order in ε the KdV equation

$$A_\tau = \pm\left(\tfrac{1}{6} - \tfrac{b}{2}\right)A_{\xi\xi\xi} \pm \tfrac{3}{2} A A_\xi \tag{11}$$

with $\tau = \varepsilon^3 t, \xi = \varepsilon(x \pm t)$.

- For $b = \frac{1}{3} + 2\nu\varepsilon^2$ one gets, by making the ansatz

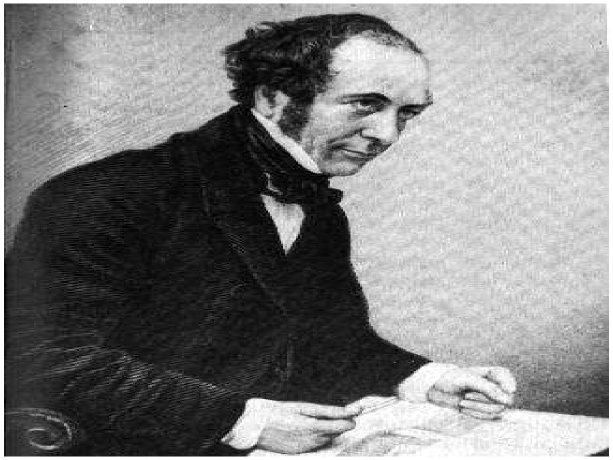
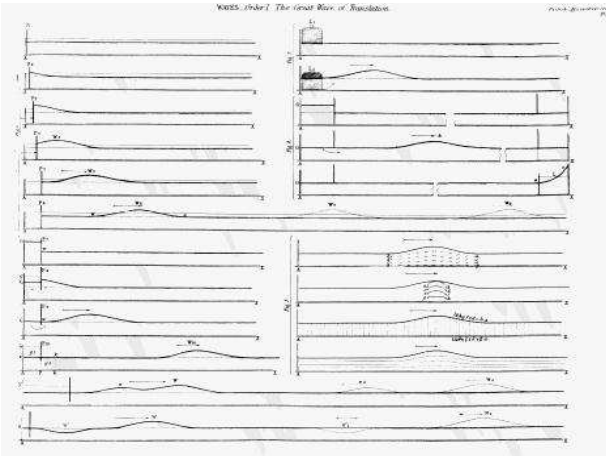
$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix}(x,t) = \varepsilon^4 A\left(\varepsilon(x \pm t), \varepsilon^5 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathscr{O}(\varepsilon^5)$$

the Kawahara equation

$$\partial_\tau A = \mp \nu \partial_\xi^3 A \pm \tfrac{1}{90}\partial_\xi^5 A \pm \tfrac{3}{2} A \partial_\xi A \tag{12}$$

with $\tau = \varepsilon^5 t, \xi = \varepsilon(x \pm t)$.

- Consequently, the soliton dynamics of the KdV equation and the dynamics of the Kawahara equation are at least approximately present in the 2-d water wave problem.
- Solitons were first observed experimentally by John Scott Russell in 1834 (J. S. Russell: Report on waves. Rep. 14th Meet. Brit. Assoc. Adv. Sci., York, London, John Murray, (1844), 311–390).


• Consequence of solitary waves: speed limits for high speed ferries

(a) HSC operating at sub-critical speed in the Marlborough Sounds

(b) HSC operating near super-critical speed in the Marlborough Sound

- Rigorous justification of the KdV and the Kawahara approximation by proving that the relative error of the approximation is small on the characteristic time scale of the approximation equation.
- Previous approximation proofs on the right time scales:
  Craig (1985), Schneider–Wayne (2000, 2002) : using Lagrangian coordinates
  Bona–Colin–Lannes (2005), Iguchi (2007): using Eulerian coordinates
- We present an alternative approximation proof using the arc length formulation of the water wave problem:

  W.-P. D. Validity of the Korteweg-de Vries Approximation for the Two-Dimensional Water Wave Problem in the Arc Length Formulation. *Comm. Pure Appl. Math.* **65** (2012), no. 3, 381-429.
- We prove the following theorems:

Theorem 1:

For all $b_0, C_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ and all $b \in \mathbb{R} \setminus \{\frac{1}{3}\}$ with $0 \leq b \leq b_0$ the following is true. Let

$$\eta|_{t=0}(x) = \varepsilon^2 \Phi_1(\varepsilon x), \qquad v_1|_{t=0}(x) = \varepsilon^2 \Phi_2(\varepsilon x)$$

with $\max\{\|(\Phi_1(\cdot), \Phi_2(\cdot))\|_{H_\xi^{s+8}}, \|(\rho^k \Phi_1(\cdot), \rho^k \Phi_2(\cdot))\|_{H_\xi^{s+3}}\} \leq C_0 \varepsilon^l$, where $\xi = \varepsilon x$, $s \geq 7$, $k > 1$, $l \geq 0$ and $\rho(\xi) = (1+\xi^2)^{1/2}$. Let

$$(A_1)_\tau = \left(\tfrac{b}{2} - \tfrac{1}{6}\right)(A_1)_{\xi\xi\xi} - \tfrac{3}{2} A_1 (A_1)_\xi, \qquad (A_2)_\tau = \left(\tfrac{1}{6} - \tfrac{b}{2}\right)(A_2)_{\xi\xi\xi} + \tfrac{3}{2} A_2 (A_2)_\xi,$$

$$A_1|_{\tau=0} = \tfrac{1}{2}(\Phi_1 + \Phi_2), \qquad A_2|_{\tau=0} = \tfrac{1}{2}(\Phi_1 - \Phi_2).$$

Then there is a unique solution of (8)–(9) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_0/\varepsilon^3]} \left\| \begin{pmatrix} \eta \\ v_1 \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H_\xi^s \times H_\xi^{s-1/2}} \lesssim \varepsilon^{4+l}$$

where

$$\psi(x,t) = \varepsilon^2 A_1\left(\varepsilon(x-t), \varepsilon^3 t\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2\left(\varepsilon(x+t), \varepsilon^3 t\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Theorem 2:

Let $b = \frac{1}{3} + 2\nu\varepsilon^2$. For all $\tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ the following is true. Let

$$\eta|_{t=0}(x) = \varepsilon^4 \Phi_1(\varepsilon x), \qquad u_1|_{t=0}(x) = \varepsilon^4 \Phi_2(\varepsilon x)$$

with $\|(\Phi_1, \Phi_2)\|_{H_\xi^{s+10} \cap H_\xi^{s+3}(k)} \lesssim \varepsilon^l$, where $\xi = \varepsilon x$, $s \geq 7$, $k > 1$ and $l \geq 0$. Let

$$(A_1)_\tau = \nu \partial_\xi^3 A_1 - \frac{1}{90}\partial_\xi^5 A_1 - \frac{3}{2} A_1 \partial_\xi A_1, \qquad (A_2)_\tau = -\nu \partial_\xi^3 A_2 + \frac{1}{90}\partial_\xi^5 A_2 + \frac{3}{2} A_2 \partial_\xi A_2,$$

$$A_1|_{\tau=0} = \frac{1}{2}(\Phi_1 + \Phi_2), \qquad A_2|_{\tau=0} = \frac{1}{2}(\Phi_1 - \Phi_2).$$

Let $[0, \tau_1]$ be the existence interval of A_1, A_2 in $H_\xi^{s+10} \cap H_\xi^{s+3}(k)$ and $\tau_2 = \min\{\tau_0; \tau_1\}$. Then there is a unique solution of (8)–(9) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_2/\varepsilon^5]} \left\| \begin{pmatrix} \eta \\ u_1 \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H_\xi^s \times H_\xi^{s-1/2}} \lesssim \varepsilon^{6+l}$$

where

$$\psi(x,t) = \varepsilon^4 A_1\big(\varepsilon(x-t), \varepsilon^5 t\big) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^4 A_2\big(\varepsilon(x+t), \varepsilon^5 t\big) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Main advantages of using the arc length formulation:
  - Proof of the local–wellposedness (Ambrose–Masmoudi) is more elementary and less complex than in Eulerian or in Lagrangian coordinates,
  - Better regularity properties,
  - Our error estimates for the KdV approximation are the only ones being uniform w.r.t. the strength of the surface tension as b and ε go to 0,
  - Therefore, the cases with and without surface tension can be handled together in one approximation proof,
  - Optimal powers of ε in our bounds for the error and its spatial derivatives,
  - Arc length parametrization in the KdV– and the Kawahara–regime is close to Eulerian coordinates,
  - Therefore, transferring our error estimates into Eulerian coordinates does not weaken the estimates,
  - More accessible to generalizations, e.g., including vorticity (in preparation).

4. The 2-d Water Wave Equations and the KdV Approximation in the Arc Length Formulation

- Let $P(t) : \mathbb{R} \to \Gamma(t)$, $\alpha \mapsto P(\alpha,t) = (x(\alpha,t), y(\alpha,t))$ be a parametrization of the top surface by arc length.
- Then we have:

$$(x,y)_t = U\hat{n} + T\hat{t}, \tag{13}$$

  where U and T are the normal and the tangential velocity of Γ w.r.t. P.
- T is determined (up to a constant) by the arc length condition, which yields

$$T = \int \theta_\alpha U \,, \tag{14}$$

  where $\theta = \arctan(y_\alpha/x_\alpha)$ is the tangent angle.
- The irrotationality of the flow and the Neumann boundary condition (5) imply that for known $\Gamma(t)$ the normal velocity $U(t)$ is uniquely determined by the Lagrangian tangential velocity $v(t)$ via the so–called Birkhoff–Rott integral W.

- The Birkhoff–Rott integral $W : (\gamma, \sigma) \mapsto W(\gamma, \sigma) = W_1(\gamma) + W_2(\sigma)$ is defined by

$$(\mathrm{Re}\, W_1(\gamma) - i\, \mathrm{Im}\, W_1(\gamma))(\alpha, t) = \frac{1}{2\pi i} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\gamma(\alpha', t)}{z(\alpha, t) - z(\alpha', t)} d\alpha', \tag{15}$$

$$(\mathrm{Re}\, W_2(\sigma) - i\, \mathrm{Im}\, W_2(\sigma))(\alpha, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma(\alpha', t)}{z(\alpha, t) - z_B(\alpha')} d\alpha', \tag{16}$$

  where z and z_B are complex representations of arc length parametrizations of the top surface and the bottom. γ is the so–called vortex sheet strength and σ is the so–called source strength.

- σ is uniquely determined by γ via

$$\sigma(\alpha', t) = \frac{1}{\pi} \mathrm{Re} \int_{-\infty}^{\infty} \frac{\gamma(\alpha'', t)}{z(\alpha, t) - z_B(\alpha')} d\alpha''. \tag{17}$$

- γ is uniquely determined by v via

$$\tfrac{1}{2}\gamma + W(\gamma) \cdot \hat{t} = v. \tag{18}$$

- Finally, we have

$$U = W \cdot \hat{n}. \tag{19}$$

- v is governed by the incompressible Euler equations (1)–(2) and the boundary conditions (4)–(5). This yields

$$v_t = -y_\alpha + b\theta_{\alpha\alpha} - (v-T)(v-T)_\alpha + (W\cdot\hat{n})\theta_t\,. \tag{20}$$

- Finally, the 2-d water wave equations are equivalent to the following evolutionary system:

$$\begin{aligned} y_t &= (W\cdot\hat{n})cos\theta + Ty_\alpha\,, & (21)\\ v_t &= -y_\alpha + b\theta_{\alpha\alpha} - \delta\delta_\alpha + (W\cdot\hat{n})\theta_t\,, & (22)\\ \theta_t &= W_\alpha\cdot\hat{n} + \tfrac{\gamma}{2}\theta_\alpha - \delta\theta_\alpha\,, & (23)\\ \delta_{\alpha t} &= -(1+c)\theta_\alpha + b\theta_{\alpha\alpha\alpha} - (\delta\delta_\alpha)_\alpha + \left(W_\alpha\cdot\hat{n} + \tfrac{\gamma}{2}\theta_\alpha\right)^2, & (24)\end{aligned}$$

where

$$\delta = v - T\,, \tag{25}$$

$$c = W_t\cdot\hat{n} + \delta(W_\alpha\cdot\hat{n}) + \tfrac{\gamma}{2}\theta_t + \tfrac{\gamma}{2}\delta\theta_\alpha + (cos\theta - 1)\,. \tag{26}$$

- We will use the equations for y and v to perform the approximation and the equations for θ and δ_α to estimate the derivatives of the error.

- Now, we go to the KdV–scaling:

$$\alpha = \varepsilon^{-1}\underline{\alpha}, \quad y(\alpha,t) = \varepsilon^2\tilde{y}(\underline{\alpha},t), \quad v(\alpha,t) = \varepsilon^2\tilde{v}(\underline{\alpha},t)$$

and therefore

$$x(\alpha,t) = \alpha + \varepsilon^5\tilde{x}(\underline{\alpha},t), \quad z(\alpha,t) = \alpha + \varepsilon^2\tilde{z}(\underline{\alpha},t), \quad \theta(\alpha,t) = \varepsilon^3\tilde{\theta}(\underline{\alpha},t),$$

$$\gamma(\alpha,t) = \varepsilon^2\tilde{\gamma}(\underline{\alpha},t), \quad W(\alpha,t) = \varepsilon^2\tilde{W}(\underline{\alpha},t), \quad \delta(\alpha,t) = \varepsilon^2\tilde{\delta}(\underline{\alpha},t),$$

$$U(\alpha,t) = \varepsilon^2\tilde{U}(\underline{\alpha},t), \quad T(\alpha,t) = \varepsilon^5\tilde{T}(\underline{\alpha},t).$$

- In this scaling the system becomes

$$\tilde{y}_t = (\tilde{W}\cdot\hat{n})(1+(cos(\varepsilon^3\tilde{\theta})-1)) + \varepsilon^6\tilde{T}\tilde{y}_{\underline{\alpha}}\,, \tag{27}$$

$$\tilde{v}_t = -\varepsilon\tilde{y}_{\underline{\alpha}} + \varepsilon^3 b\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{\delta}_{\underline{\alpha}} + \varepsilon^3(\tilde{W}\cdot\hat{n})\tilde{\theta}_t\,, \tag{28}$$

$$\tilde{\theta}_t = \tilde{W}_{\underline{\alpha}}\cdot\hat{n} + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_{\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{\theta}_{\underline{\alpha}}\,, \tag{29}$$

$$\tilde{\delta}_{\underline{\alpha}t} = -\varepsilon(1+\varepsilon^2\tilde{c})\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3 b\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3(\tilde{\delta}\tilde{\delta}_{\underline{\alpha}})_{\underline{\alpha}} + \varepsilon^3\left(\tilde{W}_{\underline{\alpha}}\cdot\hat{n} + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_{\underline{\alpha}}\right)^2, \tag{30}$$

where

$$\tilde{c} = \tilde{W}_t\cdot\hat{n} + \varepsilon^3\tilde{\delta}(\tilde{W}_{\underline{\alpha}}\cdot\hat{n}) + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_t + \varepsilon^6\tfrac{\tilde{\gamma}}{2}\tilde{\delta}\tilde{\theta}_{\underline{\alpha}} + (cos(\varepsilon^3\tilde{\theta})-1)\,, \tag{31}$$

$$\tfrac{1}{2}\tilde{\gamma} + \tilde{W}\cdot\hat{t} = \tilde{v}\,, \tag{32}$$

$$\tilde{\delta} = \tilde{v} - \varepsilon^3\tilde{T}\,. \tag{33}$$

Theorem 3: (Theorem 1 in the arc length formulation)

For all $b_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ and all $b \in \mathbb{R} \setminus \{\frac{1}{3}\}$ with $0 \leq b \leq b_0$ the following is true. Let

$$\tilde{y}|_{t=0}(\underline{\alpha}) = \tilde{\Phi}_1(\underline{\alpha}), \qquad \tilde{v}|_{t=0}(\underline{\alpha}) = \tilde{\Phi}_2(\underline{\alpha})$$

with $\|(\tilde{\Phi}_1, \tilde{\Phi}_2)\|_{H^{s+8}_{\underline{\alpha}} \cap H^{s+3}_{\underline{\alpha}}(k)} \lesssim \varepsilon^l$, where $s \geq 7$, $k > 1$ and $l \geq 0$. Let

$$(A_1)_\tau = \left(\tfrac{b}{2} - \tfrac{1}{6}\right)(A_1)_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} - \tfrac{3}{2} A_1 (A_1)_{\underline{\alpha}}, \qquad (A_2)_\tau = \left(\tfrac{1}{6} - \tfrac{b}{2}\right)(A_2)_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} + \tfrac{3}{2} A_2 (A_2)_{\underline{\alpha}},$$

$$A_1|_{\tau=0} = \tfrac{1}{2}(\tilde{\Phi}_1 + \tilde{\Phi}_2), \qquad A_2|_{\tau=0} = \tfrac{1}{2}(\tilde{\Phi}_1 - \tilde{\Phi}_2).$$

Then there exists a unique solution of the 2-d water wave equations (27)–(30) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_0/\varepsilon^3]} \left\| \begin{pmatrix} \tilde{y} \\ \tilde{v} \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H^s_{\underline{\alpha}} \times H^{s-1/2}_{\underline{\alpha}}} \lesssim \varepsilon^{2+l}$$

where

$$\psi(\underline{\alpha}, t) = A_1(\underline{\alpha} - \varepsilon t, \varepsilon^3 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2(\underline{\alpha} + \varepsilon t, \varepsilon^3 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

5. Main Ideas of the Proof

- *Step 1:* Find explicit expressions for the linear and the quadratic terms of the system (27)–(30) and bounds for the cubic and higher order terms.
  Main tools: 1. Taylor expansions of the Birkhoff–Rott Integral and its derivatives. 2. Find the right balance between size and regularity.
- *Step 2:* Refind the KdV equation approximately in (27)–(30).
- *Step 3:* Write the exact solutions of (27)–(30) as approximation plus error and construct a suitable nonlinear energy being equivalent to the square of a Sobolev–norm to estimate the error on a timespan of order $\mathscr{O}(\varepsilon^{-3})$ w.r.t. t.
- *Step 4:* Express the proven result in Eulerian coordinates.
- Treat the Kawahara case analogously.

- To Step 1: We obtain

$$
\begin{aligned}
\tilde{y}_t &= K_0(\tilde{v}) + \varepsilon^3\big(K_0[K_0,\tilde{y}]\tilde{v} - (1+K_0^2)(\tilde{y}\tilde{v})\big)_{\underline{\alpha}} + h.o.t.,\\
\tilde{v}_t &= -\varepsilon\tilde{y}_{\underline{\alpha}} + \varepsilon^3 b\,\tilde{y}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{v}_{\underline{\alpha}} + \varepsilon^3 K_0(\tilde{\delta}_{\underline{\alpha}})K_0(\tilde{v}) + h.o.t.,\\
\tilde{\theta}_t &= K_0(\tilde{\delta}_{\underline{\alpha}}) - \varepsilon^3\tilde{\delta}\,\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3\big(K_0[K_0,\tilde{y}]\tilde{\delta}_{\underline{\alpha}} - (1+K_0^2)(\tilde{y}\tilde{\delta}_{\underline{\alpha}})\big)_{\underline{\alpha}}\\
&\quad + \varepsilon^3\big(K_0[K_0,\tilde{\theta}]\tilde{\delta}_{\underline{\alpha}} - (1+K_0^2)(\tilde{\theta}\tilde{\delta}_{\underline{\alpha}})\big) + h.o.t.,\\
\tilde{\delta}_{\underline{\alpha}t} &= -\varepsilon\big(1 - \varepsilon^3 K_0(\tilde{\theta}) + \varepsilon^5 b\,K_0(\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}}) + h.o.t.\big)\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3 b\,\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}}\\
&\quad - \varepsilon^3(\tilde{\delta}\tilde{\delta}_{\underline{\alpha}})_{\underline{\alpha}} + \varepsilon^3(K_0(\tilde{\delta}_{\underline{\alpha}}))^2 + h.o.t.,\\
\tilde{\delta}(\underline{\alpha},t) &= \tilde{v}(\underline{\alpha},t) - \varepsilon^3\int_{-\infty}^{\underline{\alpha}}(K_0(\tilde{v})\tilde{\theta}_{\underline{\alpha}})(\underline{\beta},t)\,d\underline{\beta} + h.o.t.,
\end{aligned}
$$

where

$$\hat{K}_0(\underline{k}) = -i\tanh(\varepsilon\underline{k})\,.$$

- To Step 3: Let

$$
\begin{aligned}
\tilde{y}(\underline{\alpha},t) &= A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)+A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{y}}(\underline{\alpha},t)\,,\\
\tilde{v}(\underline{\alpha},t) &= A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)-A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{v}}(\underline{\alpha},t)\,,\\
\tilde{\theta}(\underline{\alpha},t) &= \partial_{\underline{\alpha}}A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)+\partial_{\underline{\alpha}}A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{\theta}}(\underline{\alpha},t)\,,\\
\tilde{\delta}_{\underline{\alpha}}(\underline{\alpha},t) &= \partial_{\underline{\alpha}}A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)-\partial_{\underline{\alpha}}A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{\delta}_{\underline{\alpha}}}(\underline{\alpha},t)\,.
\end{aligned}
$$

Then the error $R=(R_{\tilde{y}},R_{\tilde{v}},R_{\tilde{\theta}},R_{\tilde{\delta}_{\underline{\alpha}}})$ satisfies

$$
\begin{aligned}
\partial_t R_{\tilde{y}} &= K_0 R_{\tilde{v}}+\varepsilon^3\mathscr{N}_1\,,\\
\partial_t R_{\tilde{v}} &= -\varepsilon\,\partial_{\underline{\alpha}}R_{\tilde{y}}+\varepsilon^3 b\,\partial_{\underline{\alpha}}^3 R_{\tilde{y}}+\varepsilon^3\mathscr{N}_2\,,\\
\partial_t R_{\tilde{\theta}} &= K_0 R_{\tilde{\delta}_{\underline{\alpha}}}-\varepsilon^3\tilde{\delta}\,\partial_{\underline{\alpha}}R_{\tilde{\theta}}-\varepsilon^3\partial_{\underline{\alpha}}(1+K_0^2)(\tilde{y}R_{\tilde{\delta}_{\underline{\alpha}}})+\varepsilon^3\mathscr{N}_3\,,\\
\partial_t R_{\tilde{\delta}_{\underline{\alpha}}} &= -\varepsilon(1+\varepsilon^3 C_R)\,\partial_{\underline{\alpha}}R_{\theta}+\varepsilon^3 b\,\partial_{\underline{\alpha}}^3 R_{\tilde{\theta}}-\varepsilon^6 b\,(\partial_{\underline{\alpha}}\tilde{\theta})K_0\partial_{\underline{\alpha}}^2 R_{\tilde{\theta}}\\
&\quad -\varepsilon^3\tilde{\delta}\,\partial_{\underline{\alpha}}R_{\tilde{\delta}_{\underline{\alpha}}}+\varepsilon^3\mathscr{N}_4\,.
\end{aligned}
$$

- We use the following energy:

$$\mathscr{E}(t) = E(t) + E_b(t) + \sum_{k=0}^{s} E_k(t) + \sum_{k=0}^{s} E_{b,k}(t)$$

for $s \geq 6$, where

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} R_{\tilde{y}} K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}}) R_{\tilde{y}}\, d\underline{\alpha} + \frac{1}{2}\int_{\mathbb{R}} R_{\tilde{v}}^2\, d\underline{\alpha},$$

$$E_b(t) = \frac{b}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}} R_{\tilde{y}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(\partial_{\underline{\alpha}} R_{\tilde{y}})\, d\underline{\alpha},$$

$$E_k(t) = \frac{1}{2}\int_{\mathbb{R}} (1+\varepsilon^3 C_R)(\partial_{\underline{\alpha}}^k R_{\tilde{\theta}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(\partial_{\underline{\alpha}}^k R_{\tilde{\theta}})\, d\underline{\alpha} + \frac{1}{2}\int_{\mathbb{R}} (\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}})^2\, d\underline{\alpha}$$

$$+\frac{1}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(1+K_0^2)(\tilde{y}\,\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}})\, d\underline{\alpha},$$

$$E_{b,k}(t) = \frac{b}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}}^{k+1} R_{\tilde{\theta}})\big(K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}}) + \varepsilon^4(\partial_{\underline{\alpha}}\tilde{\theta}) + \varepsilon^6(\partial_{\underline{\alpha}} R_{\tilde{\theta}})\big)(\partial_{\underline{\alpha}}^{k+1} R_{\tilde{\theta}})\, d\underline{\alpha}.$$

- We show:
$$\frac{d}{dt}\mathscr{E} \lesssim \varepsilon^3(\mathscr{E}+1)$$
uniformly w.r.t. all $b \in \mathbb{R}_0^+ \setminus \{\frac{1}{3}\}$ with $b \le b_0$.

  Main ingredients of the argumentation:
  - The energy $\mathscr{E}$ is constructed in a such way that all terms in the error equations that cannot be estimated directly cancel.
  - Transport terms do not cause a loss of regularity.
  - Use of commutator estimates.
- Now, an application of Gronwall's inequality yields the boundedness of the error on the right time scale.

6. The NLS Approximation

- Let $b = 0$. Inserting the ansatz

$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix}(x,t) = \varepsilon\Psi_{NLS}(x,t) = \varepsilon A\left(\varepsilon(x-c_g t), \varepsilon^2 t\right) e^{i(k_0 x - \omega(k_0)t)}\varphi(k_0) + c.c. \qquad (\varepsilon \ll 1)$$

with

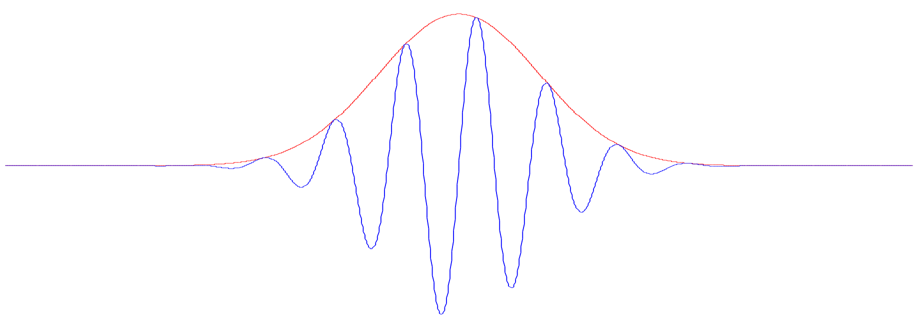
$$\omega(k) = \operatorname{sign}(k)\sqrt{k\tanh(k)}, \qquad c_g = \partial_k\omega(k_0), \qquad \varphi(k_0) \in \mathbb{C}^2$$

in the 2-d water wave equations (8)–(9) yields at leading order in ε the NLS equation

$$A_\tau \;=\; i\nu_1 A_{\xi\xi} + i\nu_2 A|A|^2 \tag{34}$$

with $\tau = \varepsilon^2 t, \xi = \varepsilon(x - c_g t)$ and $\nu_j = \nu_j(k_0) \in \mathbb{R}$ (Zakharov, 1968).

- The NLS equation describes the evolution of the envelope of an oscillating wave packet with basic wave number k_0.



- The approximation of solutions to the 2-d water wave equations with the help of the NLS equation is also interesting in the context of modeling Monster Waves.

- Rigorous justification of the NLS approximation by proving that the relative error of the approximation is small on the characteristic time scale of the approximation equation, i.e., $t = \mathscr{O}(\varepsilon^{-2})$.

- Mathematical rigorous justifications of the NLS approximation in the right scales exist in the case without surface tension
  Totz-Wu: for infinite depth of water,
  D.-Schneider-Wayne: for finite depth of water:

  W.-P. D., G. Schneider, C. E. Wayne. Justification of the Nonlinear Schrödinger equation for the evolution of gravity driven 2D surface water waves in a canal of finite depth. *Arch. Rat. Mech. Anal.* (2015), http://dx.doi.org/10.1007/s00205-015-0937-z

- We prove the following theorem:

Theorem 4:

Let $b = 0$ and $s \geq 7$. Then for all $k_0 > 0$ and for all $C_1, \tau_0 > 0$ there exist $\tau_1 > 0$ and $\varepsilon_0 > 0$ such that for all solutions $A \in C([0,\tau_0], H^s(\mathbb{R},\mathbb{C}))$ of the NLS equation (34) with

$$\sup_{\tau \in [0,\tau_0]} \|A(\cdot,\tau)\|_{H^s(\mathbb{R},\mathbb{C})} \leq C_1$$

the following holds. For all $\varepsilon \in (0,\varepsilon_0)$ there exists a solution

$$(\eta, v_1) \in C([0,\tau_1/\varepsilon^2], (H^s(\mathbb{R},\mathbb{R}))^2)$$

of the 2-D water wave problem (8)–(9) which satisfies

$$\sup_{t \in [0,\tau_1/\varepsilon^2]} \left\| \begin{pmatrix} \eta \\ v_1 \end{pmatrix} (\cdot,t) - \psi(\cdot,t) \right\|_{(H^s(\mathbb{R},\mathbb{R}))^2} \lesssim \varepsilon^{3/2}$$

where

$$\psi(x,t) = \varepsilon A\left(\varepsilon(x - c_g t), \varepsilon^2 t\right) e^{i(k_0 x - \omega(k_0)t)} \varphi(k_0) + c.c..$$

- A rigorous justification of the NLS approximation in the right scales in the case with surface tension exists at the moment only for the Boussinesq equation

$$\partial_t^2 u = \partial_x^2 u + \partial_x^2 (u^2) + \partial_t^2 \partial_x^2 u + \mu \partial_x^6 u \,,$$

which is another reduced model equation for the 2-d water wave equations.

W.-P. D., G. Schneider. Justification of the nonlinear Schrödinger equation for a resonant Boussinesq model. *Indiana Univ. Math. J.* **55** (2006), no. 6, 1813-1834.

- Moreover, we have rigorous justifications of the NLS approximation in the right scales for the quasilinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u - u^3 - \partial_x^2 (u^2) \,.$$

M. Chirilus-Bruckner, W.-P. D., G. Schneider. NLS approximation of time oscillatory long waves for equations with quasilinear quadratic terms. *Math. Nachr.* **288** (2015), no. 2-3, 158-166.

and for the 2D fourth order nonlinear wave equation

$$\partial_t^2 u = \Delta u - u - \Delta^2 u + u^2 \,.$$

W.-P. D., A. Hermann, G. Schneider, D. Zimmermann. Justification of the 2D NLS equation – Quadratic resonances do not matter in case of analytic initial conditions. Submitted to *J. Math. Anal. Appl.*, 2015.

- Very recently, we have rigorously justified the NLS approximation in the right scales for the quasilinear dispersive equation

$$\partial_t u = K_0 u - \partial_x(u^2)\,.$$

W.-P. D., M. Heß. Validity of the Nonlinear Schrödinger approximation for a quasilinear dispersive equation. Preprint, Universität Stuttgart. In Preparation. 2015

- Slow modulations of non-decaying periodic traveling wave solutions to the 2-d water wave equations can be formally approximated by Whitham's equations (a nonlinear system of hyperbolic conservation laws).

- The mathematical rigorous justification of the Whitham approximation is an open problem.

- A first step: Justification of the Whitham approximation for a spatially periodic Boussinesq model.

  R. Bauer, W.-P. D., G. Schneider. The KdV, the Burgers and the Whitham limit for a spatially periodic Boussinesq model. Preprint, Universität Stuttgart, 2015.

7. Main Ideas of the Proof of Theorem 4

- Step 1: Using Lagrangian coordinates

$$\Gamma(t) = \{X(\alpha,t) = (\tilde{X}_1(\alpha,t), \tilde{X}_2(\alpha,t)) = (\alpha + X_1(\alpha,t), X_2(\alpha,t) \,|\, \alpha \in \mathbb{R}\} \ .$$

- In terms of the variables

$$\mathscr{W} = (Z_1 = \mathscr{K}_0 X_1,\ X_2,\ U_1 = \partial_t X_1),$$

where

$$\hat{\mathscr{K}_0}(k) = -i \tanh(k),$$

the evolution equations are

$$\partial_t \mathscr{W} = \begin{pmatrix} \mathscr{K}_0 U_1 \\ \mathscr{K}_0 U_1 + \mathscr{S}_1(X) U_1 \\ -(1 - \mathscr{M}_2 Z_1 + (\partial_\alpha X_2)\mathscr{K}_0 + (\partial_\alpha X_2)\mathscr{S}_1(X))^{-1}((\partial_\alpha X_2)(1 + [\partial_t, \mathscr{S}_1(X)]U_1)) \end{pmatrix},$$

where

$$\mathscr{S}_1(X) = \mathscr{K}(X) - \mathscr{K}_0\,, \quad \mathscr{M}_2 = -\partial_\alpha (\mathscr{K}_0)^{-1}\,.$$

- Step 2: After diagonalizing the linear part of this system the equations are of the form

$$\partial_t c \;=\; \Lambda c + N(c,c)\,, \tag{35}$$

where

$$c = (c_1, c_2, c_3)^T\,, \quad \Lambda = i\,\mathrm{diag}(\omega_1, \omega_2, \omega_3)$$

with

$$\omega_1(k) = 0\,, \quad \omega_2(k) = -\omega(k) = -\mathrm{sign}(k)\sqrt{k\tanh(k)}\,, \quad \omega_3(k) = \omega(k)\,.$$

- Step 3: Writing the solution as approximation and error, i.e.,

$$c = \varepsilon\Psi + \varepsilon^3 R\,,$$

where $\Psi - (0, \Psi_{NLS}, 0)^T = \mathscr{O}(\varepsilon)$ yields

$$\partial_t R \;=\; \Lambda R + 2\varepsilon N(\Psi, R) + \varepsilon^3 N(R,R) + Res(\Psi)\,, \tag{36}$$

where $Res(\Psi) = \mathscr{O}(\varepsilon^2)$ for an appropriate construction of Ψ.

- Hence, the evolution system for R is of the form

$$\partial_t R = \Lambda R + \varepsilon B(\Psi, R) + \mathscr{O}(\varepsilon^2), \tag{37}$$

where

$$\hat{B}_j(\Psi, R)(k) = \int \sum_{l,n=1,2,3} \hat{b}_{j,l,n}(k, k-\ell, \ell)\hat{\Psi}_l(k-\ell)\hat{R}_n(\ell)d\ell \;, \quad j = 1,2,3.$$

- Step 4: Since we cannot directly control $\varepsilon B(\Psi, R)$ over a time scale of order $\mathscr{O}(\varepsilon^{-2})$, we want to eliminate this term with the help of a normal form transformation $R \mapsto \tilde{R}$ with

$$\tilde{R} = R + \varepsilon M(\Psi, R) \;, \tag{38}$$

where

$$\hat{M}_j(\Psi, R)(k) = \int \sum_{l,n=1,2,3} \hat{m}_{j,l,n}(k, k-\ell, \ell)\hat{\Psi}_l(k-\ell)\hat{R}_n(\ell)d\ell \;, \quad j = 1,2,3.$$

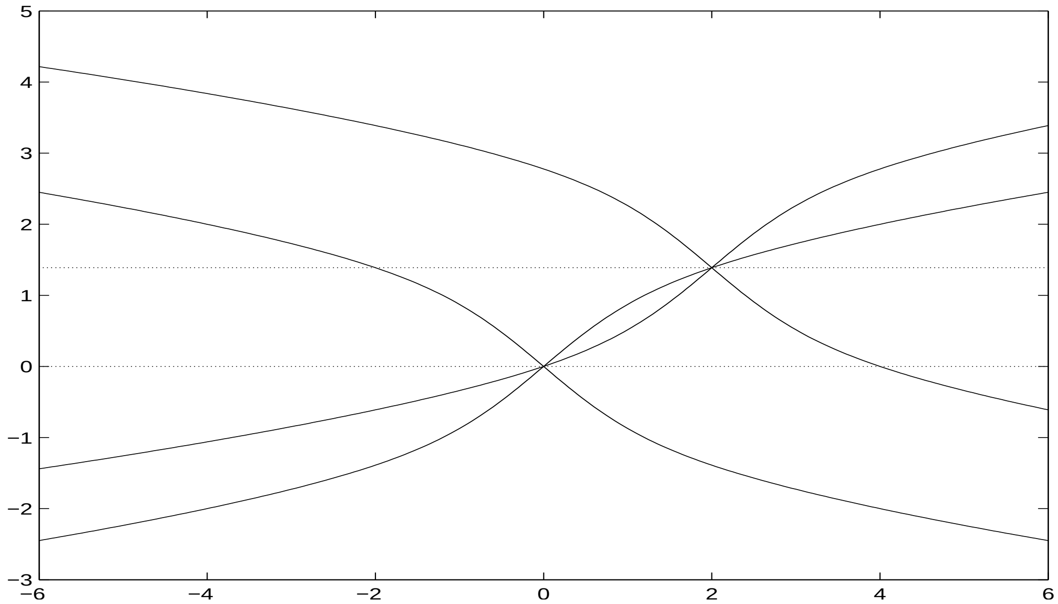
- Inserting (38) to (37) yields

$$\hat{m}_{j,l,n}(k, k-\ell, \ell) = \frac{i\hat{b}_{j,l,n}(k, k-\ell, \ell)}{-\omega_j(k) - \omega(k-\ell) + \omega_n(\ell)}.$$

- Since $\hat{\Psi}_{NLS}$ is concentrated around $k = \pm k_0$ the kernels $\hat{m}_{j,l,n}$ can be simplified to

$$\hat{m}_{j,l,n}(k) = \frac{i\hat{b}_{j,l,n}(k, \pm k_0, k \mp k_0)}{-\omega_j(k) - \omega(\pm k_0) + \omega_n(k \mp k_0)}.$$

- First difficulty: There are points where the denominator vanishes (resonances).
- The resonances are located at $k = 0, \pm k_0, \pm 2k_0$.



The curves $k \mapsto \omega_j(k)$ and the curves $k \mapsto \omega_3(k_0) + \omega_m(k - k_0)$ for $j, m \in \{1,2,3\}$ and $k_0 = 2$. Intersection points correspond to resonances.

- For the resonances at $k=0$ and $k=\pm 2k_0$ the numerators also vanish and the singularities can be removed.
- For the resonances at $k=\pm k_0$ the singularities can be removed by rescaling R, i.e., by replacing R by ϑR, where

$$\hat{\vartheta}(k)=\begin{cases} 1 & \text{for} \quad |k|>\delta\,, \\ \varepsilon+(1-\varepsilon)|k|/\delta & \text{for} \quad |k|\leq\delta\,, \end{cases} \qquad \delta\ll 1.$$

- Second difficulty: Since $\hat{b}_{j,l,n}(k, \pm k_0, k \mp k_0) \sim i\,\mathrm{sign}(k)\sqrt{|k|}$ as $|k| \to \infty$ for some $\hat{b}_{j,l,n}$, the normal form transform loses smoothness, i.e., we have

$$\|\tilde{R}\|_{H^s} \lesssim \|R\|_{H^{s+1/2}} \ .$$

- But, to obtain an evolution equation for $\tilde{R}$, we have to invert the normal form transform.
- Despite the loss of smoothness the normal form transform can be inverted, where

$$\|R\|_{H^s} \lesssim \|\tilde{R}\|_{H^s} \ .$$

- The reason for that is the fact that the mapping $R \mapsto \tilde{R}$ has similar structural properties as the solution operator $f \mapsto u$ of the differential equation

$$u(x) + \varepsilon a(x) u_x(x) = f(x) \ , \quad a(x) \in \mathbb{R}.$$

- Having made the normal form transform the error $\tilde{R}$ satisfies

$$\partial_t \tilde{R} = \Lambda \tilde{R} + \varepsilon^2 f(\tilde{R}) + Res(\Psi) \ , \tag{39}$$

  where

$$\|f(\tilde{R})\|_{H^s} \lesssim \|\tilde{R}\|_{H^{s+1}} \ .$$

- Hence, we have no more $\mathscr{O}(\varepsilon)$–terms but the RHS loses one derivative.
- Therefore, we will control the error $\tilde{R}$ in the following space of analytic functions:

$$Y_{\sigma,s} = \left\{ f \in L^2(\mathbb{R}) \mid \|f\|_{Y_{\sigma,s}} = \left(\int (1+k^2)^s e^{2\sigma|k|} |\hat{f}(k)|^2 dk \right)^{1/2} < \infty \right\}.$$

  This is possible because Ψ_{NLS} is compactly supported in Fourier space up to a small error.

- Step 5: We introduce the new variables w, where

$$\hat{\tilde{R}}(k,t) = \hat{w}(k,t)e^{-|k|(a-b\varepsilon^2 t)}$$

with an appropriate choice of $a, b > 0$.

- Then w satisfies

$$\partial_t w = \Lambda w - |k| b \varepsilon^2 w + \dots$$

- Therefore, we obtain

$$\partial_t \|w\|_{H^s} \lesssim \varepsilon^2 (\|w\|_{H^s} + 1) \ .$$

- Applying Gronwall's lemma, we get the boundedness of $\|w\|_{H^s}$ and therefore the boundedness of $\|\tilde{R}\|_{Y_{\sigma,s}}$ on a timespan of order $\mathscr{O}(\varepsilon^{-2})$.
- Since the $Y_{\sigma,s}$-norm controls any Sobolev norm, we finally obtain the boundedness of the error in H^s on a timespan of order $\mathscr{O}(\varepsilon^{-2})$, which proves our theorem.